Talbot 2022: Scissors Congruence and Algebraic K-theory

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Acknowledgements

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Warning: Many of these talks were lived TeX'd and although we have done our best to edit the content, there are possibly still typos, notational inconsistencies, or other mathematical errors.

0 Talk 0: Introduction (Inna Zakharevich)

References: [Jes68; Dup01; Syd65].

0.1 Scissors congruence

How do we assign a number to area? In ancient Greece, numbers were things which were associated to lengths, so asking for area to correspond to a number didn't really make sense. Euclid defined area as "that which does not change under decomposition." In more modern language, we have the following definition:

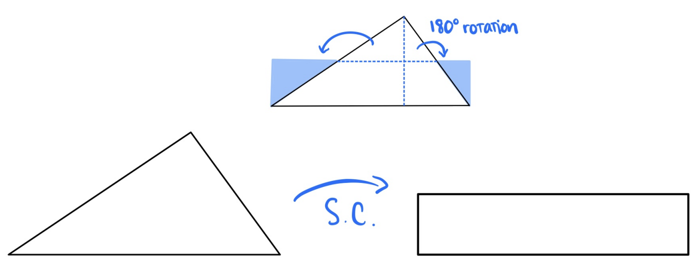
Definition 0.1 (Scissors congruence)**.** Two polygons P and Q are **scissors congruent** if $P = \dot\cup P_i$ and $Q = \dot\cup Q_i$ such that $P_i \cong Q_i$ for all i. That is, there is an isometry $g_i \in \mathrm{Isom}(E^2)$ such that $g_i P_i = Q_i$.

Notation. By $P = \dot\cup P_i$, we mean that $P = \cup P_i$ and the intersections $P_i \cap P_j$ have measure zero.

Theorem 0.2. *P and Q are scissors congruent iff* $\mathrm{area}(P) = \mathrm{area}(Q)$*.*

Remark 0.3. This theorem tells us Euclid's notion of area was well-defined. The forward implication of the theorem is not too bad so we will focus on the converse.

Proof idea ($\Leftarrow$). First note that it suffices to show that any polygon P is scissors congruent to the rectangle $1 \times \mathrm{area}(P)$. Moreover, we can triangulate any polygon so in fact it suffices to show it when P is a triangle. But we can turn any triangle into a base$\times\frac{\text{height}}{2}$ rectangle like this:



So we just need to show that any rectangle R is scissors congruent to the $1 \times \mathrm{area}(R)$ rectangle. Finishing this part is left as an exercise. ☐

Challenge. Inna collects different proofs that a rectangle R is scissors congruent to the $1 \times \mathrm{area}(R)$ rectangle, so if you come up with an interesting one, let her know.

Remark 0.4. Note that we didn't need to use all of the isometry group of E^2. Turning the triangle into a rectangle required a 180° rotation, and the rest (including the challenge/finishing the proof) can be done using only translations.

Exercise 0.5. Figure out the 0-dimensional and 1-dimensional cases.

Question 0.6 (Hilbert's 3rd Problem)**.** Is scissors congruence a well-defined notion of volume? That is, if two polyhedra[1] have the same volume, are they scissors congruent? Can we find a counterexample?

The first question to ask is how many cuts we're allowed to make. If we allow *infinitely* many cuts, then the techniques of calculus tell us yes, that two polyhedra of the same volume are scissors congruent. But what if we only allow finitely many cuts? In 1901, shortly(!) after Hilbert proposed this problem, it was answered by his student(!!) Dehn.

Theorem 0.7 (Dehn, 1901)**.** *The cube and the regular tetrahedron are not scissors congruent.*

Remark 0.8. To prove this, Dehn constructs something called the **Dehn invariant** D and shows that $D(\text{cube}) = 0$ but $D(\text{tetrahedon}) \neq 0$.

0.2 The Dehn invariant

Definition 0.9 (Dehn invariant)**.** The **Dehn invariant** of a polyhedron P is

$$D(P) = \sum_{\text{edges } e} \text{length}(e) \otimes \text{angle}(e)/\pi.$$

Remark 0.10. This invariant $D(P)$ lives in the tensor product $\mathbb{R} \otimes_{\mathbb{Z}} \mathbb{R}/\mathbb{Z}$, but tensor products were not defined until 1938! In 1965, [Syd65] showed that the Dehn invariant and volume completely characterize scissors congruence.

Theorem 0.11 (Sydler, 1965)**.** *If* $\text{vol}(P) = \text{vol}(Q)$ *and* $D(P) = D(Q)$ *then* P *is scissors congruent to* Q.[2]

Definition 0.12 (The polytope algebra)**.** Let X be a geometry, usually hyperbolic $\mathbb{H}^n$, spherical $\mathbb{S}^n$, or Euclidean $\mathbb{E}^n$, and let G be a group of isometries, for example $G = \text{Isom}(X)$). The **polytope algebra** is defined as

$$\mathcal{P}(X, G) := \mathbb{Z}[\text{Polytopes in } X]/\sim$$

where $[P \cup Q] \sim [P] + [Q]$ and $[P] \sim [gP]$ for all $g \in G$.

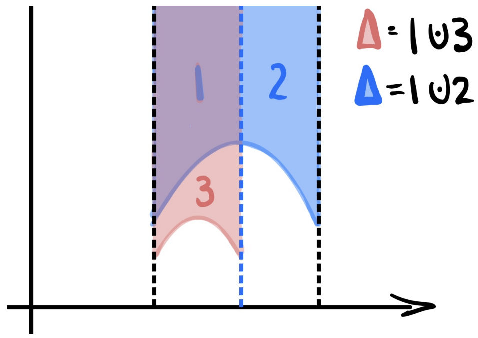
Remark 0.13. The first relation lets us decompose elements into smaller pieces and the second relation lets us consider isomorphism classes under the group action.

Theorem 0.14. *If* X *is Euclidean, spherical, or hyperbolic, then* P *and* Q *are scissors congruent iff* $[P] = [Q]$ *in* $\mathcal{P}(X, G)$.

[1] 3-dimensional polytopes.

[2] A great reference for this is Jessen's 1968 paper [Jes68]; he reframes this as a group theoretic problem.

Example 0.15 (Non-example)**.** This does not hold if X is taken to be the hyperbolic plane along with *ideal points*:



In the picture above, the red triangle can be reached from the blue triangle by an element of the isometry group (where both triangles have their third vertex the ideal point at infinity), and so the decomposition into pieces $1, 2, 3$ above implies $[2] = [3]$ in $\mathcal{P}(X, G)$. However, 2 cannot be scissors congruent to 3 because no isometry will move the ideal vertices of 2 to the non-ideal vertices of 3!

Question 0.16 (Generalized Hilbert's 3rd Problem)**.** Can we understand $\mathcal{P}(X, G)$ for $X = \mathbb{E}^n, \mathbb{S}^n, \mathbb{H}^n$ and $G \leq \mathrm{Isom}(X)$ a subgroup of isometries?

0.3 Goncharov's conjecture

In dimensions 3 and 4, we mostly understand what is going on (except maybe the hyperbolic setting), but in general for dimensions ≥ 5, this is completely unknown. However, Goncharov had a conjecture which brings K-theory into the picture, and we will state the special case for $\mathbb{H}^{2n+1}$.

Conjecture 0.17 (Goncharov's Conjecture)**.** In dimension $2n+1$, we have $2n$ Dehn invariants D_i. We can intersect their kernels and map into $\mathbb{R}$ via volume,

$$\bigcap_{i=1}^{n} \ker D_i \xrightarrow{\text{vol}} \mathbb{R}.$$

If the Dehn invariants and volume tell us everything about scissors congruence, the volume map should be injective. Goncharov conjectured that the volume factored through a somewhat mysterious group:

$$\begin{tikzcd}
\bigcap_{i=1}^{n} \ker D_i \arrow[rr, hook, "f"] \arrow[dr, "\text{vol}"'] & & (\mathrm{gr}^{\gamma}_{n+1}\mathsf{K}_{2n+1}(\mathbb{C}) \otimes \varepsilon(n+1))^{-} \arrow[dl, "B_r"] \\
& \mathbb{R} &
\end{tikzcd}$$

where B_r is the **Borel regulator**. We have have the following conjectures concerning this diagram:

- The map f exists[3],
- f is injective, and
- The Borel regulator B_r is injective for $\mathbb{C}$.

The Borel regulator is injective in many cases, e.g. if $\mathbb{C}$ is replaced with a number field, although injectivity over $\mathbb{C}$ is currently unknown.

We can say a bit more about the mystery group appearing as the image of f: the ε is a twisting factor; $\mathrm{gr}^{\gamma}_{n+1}$ is the grading that comes from the γ-filtration, which is conjecturally the same as the *rank filtration* which corresponds to picking out the dimension, and the superscript $(\cdots)^{-}$ denotes taking the -1-eigenspace under the map induced by complex conjugation. Computational evidence indicates that these conjectures are reasonable and stand a chance of being true.

0.4 Some ideas behind K-theory

Remark 0.18. The formula for scissors congruence naturally leads us into the world of algebraic K-theory, where we have the definition for any (commutative) ring R with unit

$$\mathsf{K}_0(R) \coloneqq \mathbb{Z}[\text{finitely generated projective } R\text{-modules}]/\sim$$

where $[B] = [A] + [C]$ for every exact sequence $A \hookrightarrow B \twoheadrightarrow C$. In particular, notice that this implies $[A] = [A']$ if $A \cong A'$, so this should be reminiscent of Definition 0.12.

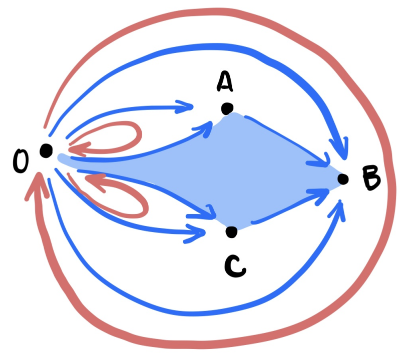
The goal of K-theory is to construct groups $\mathsf{K}_n(R)$ for all $n \geq 0$ which contain useful information about R. We can see already that K_0 is not enough: all fields are the same to K_0 since $\mathsf{K}_0(F) \cong \mathbb{Z}$ for any field F. We can start to detect the difference at the level of K_1, since $\mathsf{K}_1(F) \cong F^{\times}$, and of course $F^{\times}$ will look very different for different fields.

A breakthrough came when Quillen constructed a space (really, a spectrum) $\mathsf{K}(R)$ and defines $\mathsf{K}_n(R) \coloneqq \pi_n \mathsf{K}(R)$. To construct this space, we make two observations:

- $\mathsf{K}_0(R)$ has a three-term relation $[B] = [A] + [C]$ which is induced by a

[3]There is a more general formulation where the index on the K-group can vary.

triangle instead of just an arc:



That is, K_0 is built from *loops* and hence should be realized as π_1 of something. Thus our K-theory space should be a loop space.

- Our construction relies on the existence of two notions of "smallness." For example, kernels and cokernels or inclusions and quotients.

Definition 0.19 (The Q construction)**.** The Q-construction on the exact category $\mathcal{C} = \mathsf{Mod}^{\text{f.g.}}_{\text{proj}}(R)$ of finitely generated projective R-modules gives a new category $Q\mathcal{C}$ whose objects are the objects of $\mathcal{C}$ and morphisms generated by monics and epi-ops in $\mathcal{C}$. This encodes the two ways for one object to be "smaller" than another object in $\mathcal{C}$. Composition is given by pullback:

$$\begin{tikzcd}
 & X \times_Z Y \arrow[dl, twoheadrightarrow] \arrow[dr, hookrightarrow] & \\
X \arrow[rr, bend left, color=red] \arrow[rr, bend right, color=blue] \arrow[dr, hookrightarrow] & & Y \arrow[dl, twoheadrightarrow] \\
 & Z &
\end{tikzcd}$$

Note that for all objects A, we have a loop $0 \hookrightarrow A \twoheadleftarrow 0$ and for every exact sequence $A \hookrightarrow B \twoheadrightarrow C$ we get the picture above.

Definition 0.20 (Q-construction of K-theory)**.**

$$\mathsf{K}(R) := \mathrm{B}Q\mathcal{C}.$$

Upshot: Quillen is brilliant.

Remark 0.21. With scissors congruence, there is only *one* way of being smaller: we have inclusions but not quotients. We introduce a second way for P to be a subobject of Q:

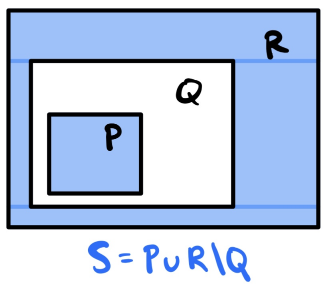
$$(1)\ P \hookrightarrow Q \quad \text{and} \quad (2)\ P \color{blue}{\hookrightarrow} \color{black} Q,$$

noting that the only difference is that the arrows are now decorated with different colors. These arrows now live in different worlds and cannot be composed.

The role previously played by kernels and cokernels is now played by *complements*. We have a commuting rule for the black and blue arrows:

$$\begin{tikzcd} P \cap Q \arrow[r, hook] \arrow[d, hook] & P \arrow[d, hook] \\ Q \arrow[r, hook] & P \cup Q \end{tikzcd}$$

Example 0.22. If we have a chain of inclusions $P \hookrightarrow Q \hookrightarrow R$, then this is the same as $P \hookrightarrow S \hookrightarrow R$ for $S = (R \setminus Q) \cup P$:



That is, we can swap the colors of the arrows by replacing Q with the object S whose union (resp. intersection) with Q is R (resp. P).

We can record this data in the category of n-polytopes under isometric embeddings, and try to do K-theory on it. To do this, we develop CGW categories and a sort of "combinatorial" higher K-theory for them.

Example 0.23. If we restrict to line segments, then we can compute

$$\mathsf{K}_i(\text{segments, translations}) \cong \mathbb{R}^{\wedge^{i+1}}$$

where the wedge is taken over $\mathbb{Z}$, and

$$\mathsf{K}_i(\text{segments, isom}) \cong \begin{cases} \mathbb{R}^{\wedge^{i+1}} & i \text{ even,} \\ 0 & i \text{ odd,} \end{cases}$$

where the isometry group is comprised of translations and reflections.

1 Talk 1 & 2: Scissors Congruence: Classical and Homological Perspectives (Claire Mirocha)

References: [Jes68; Dup01; Syd65; Sah81; DS82; DPS88].

Outline of topics

1. Classical scissors congruence.
2. The Dehn invariant D. *Main reference:* [Jes68], *Secondary reference:* [Dup01; Syd65].
3. Scissors congruence as homology.
4. The Steinberg module and the Tits building. *Main reference:* [**Ch. 2**; **3**; Dup01], *Secondary reference:* [Sah81; DS82; DPS88].

1.1 Classical scissors congruence

Setup and key definitions:

- X is a topological space (usually we'll take $X = \mathbb{R}^n$, but can also be $\mathbb{H}^n, \mathbb{S}^n$)
- $I(X)$ is the group of isometries of X (note the case $I(\mathbb{R}^n) = T(n) \rtimes O(n)$, where $T(n)$ is the group of translations.)
- A geometric n-**simplex** in X is the convex hull of $(n+1)$-many points in X, denoted $\sigma = |(a_0, \dots, a_n)|$
- A **polytope** is a finite union of simplices.
- We say that a polytope P **decomposes** into P' and P'' (written $P = P' \sqcup P''$) if $P = P' \cup P''$ and the interiors of P and P'' are disjoint.
- P and P' are G-**scissors congruent** (written $P \sim_G P'$) if $P = \bigsqcup_i P_i$ and $P' = \bigsqcup_i P_i'$, and for all i, we have $P_i = g_i P_i'$ for some $g \in G$.
- The G-**scissors congruence group** $\mathcal{P}(G, X)$ is the free abelian group generated by classes $[P]$ for P a polytope in X, with the relations:
  (i) $[P] = [P'] + [P'']$ if $P = P' \sqcup P''$
  (ii) $[P] = [gP]$ for all $g \in G$

1.2 The Dehn invariant

Question 1.1 (Hilbert's 3rd Problem)**.** Does volume determine scissors congruence for polyhedra of dimension 3?

Answer 1.2. No. Dehn found an invariant D of scissors congruence which is zero for the cube and nonzero for the regular tetrahedron, regardless of volume. Furthermore, the following theorem due to Dehn-Sydler tells us that the Dehn invariant is not only necessary, but also *sufficient* (along with volume) to determine scissors congruence completely:

Theorem 1.3 (Dehn-Sydler)**.** *For two polyhedra P and P' in $\mathbb{R}^n$, if $\operatorname{vol}(P) = \operatorname{vol}(P')$ and $D(P) = D(P')$, then $P \sim P'$.*

Remark 1.4. Note that this is *not* true for $X = \mathbb{S}^n$ or $X = \mathbb{H}^n$; in fact volume may have countable image in $\mathbb{R}$, in these cases! So what is D, then?

Definition 1.5 (The Dehn invariant)**.** The **Dehn invariant** is a $\mathbb{R}\otimes_{\mathbb{Z}}\mathbb{R}/\mathbb{Z}$-valued map D on polytopes, defined by

$$P \mapsto \sum_{\text{edges } \alpha \text{ of } P} l(\alpha) \otimes_{\mathbb{Z}} \theta(\alpha)/\pi$$

where $l(\alpha)$ and $\theta(\alpha)$ are the length and the dihedral angle, respectively, of the edge α.

Proposition 1.1. *The volume map* $\operatorname{vol} : \ker(D) \to \mathbb{R}$ *is an isomorphism.*

Proof sketch. The volume map is injective; this is the content of the Dehn-Sydler theorem, because if Dehn invariants of two polytopes are equal (and in fact, both zero), then volume determines the scissors congruence class. The volume map is surjective due to the fact that we can scale polyhedra in $\mathbb{R}^n$, and this is a similarity transformation. It is a homomorphism because it is clearly additive under unions of polyhedra whose interiors are disjoint. $\square$

Proposition 1.2. *The cokernel of D is the $\mathbb{Z}$-module $\Omega^1_{\mathbb{R}/\mathbb{Z}}$ of Kähler differentials.*

Proof sketch. We're considering the exact sequence

$$\mathcal{P} \xrightarrow{D} \mathbb{R} \otimes_{\mathbb{Z}} \mathbb{R}/\mathbb{Z} \twoheadrightarrow \operatorname{coker}(D) = (\mathbb{R} \otimes_{\mathbb{Z}} \mathbb{R}/\mathbb{Z})/\operatorname{im}(D) \to 0$$

where $\mathcal{P}$ is the free abelian group generated by polyhedra. Jessen shows that maps $\mathbb{R} \otimes_{\mathbb{Z}} \mathbb{R}/\mathbb{Z} \to V$ that vanish on $\operatorname{im}(D)$, where V is an $\mathbb{R}$-vector space, are precisely those of the form

$$\phi(l \otimes \theta) = l d \frac{\sin(\theta)}{\cos(\theta)},$$

where d is a derivation $\mathbb{R} \to V$. So, in particular, the map $\mathbb{R}\otimes_{\mathbb{Z}}\mathbb{R}/\mathbb{Z} \to \operatorname{coker}(D)$ is determined by all derivations $\mathbb{R} \to (\mathbb{R}\otimes_{\mathbb{Z}}\mathbb{R}/\mathbb{Z})/\operatorname{im}(D)$, which is exactly $\Omega^1_{\mathbb{R}/\mathbb{Z}}$. See [Jes68] for details. $\square$

1.3 Scissors congruence in homological terms

Remark 1.6. Recall the G-scissors congruence group $\mathcal{P}(X, G)$ and its two relations (i) and (ii). Both of these relations can be expressed in homological terms:

Proposition 1.3. *$\mathcal{P}(X, \{1\}) \simeq H_n(C_*(X)/C_*(X)^{n-1})$, where $C_*(X)$ is the simplicial chain complex of X, with generators of $C_k(X)$ being k-simplices $(a_0, \ldots, a_k)$ for a_i a point in X, and boundary map defined by*

$$\partial : (a_0, \ldots, a_k) \mapsto \sum_{i=0}^{k} (-1)^i (a_0, \ldots, \hat{a_i}, \ldots, a_k))$$

and $C_k(X)^{n-1}$ consists of k-simplices of dimension $n-1$ or smaller.

Proof. This statement is a homological version of the relation (i) $[P] = [P'] + [P'']$ if $P = P' \sqcup P''$. To see this, first fix an orientation on X. Then, depending on whether the ordering on the vertices of a simplex σ give it an orientation that agrees or disagrees with that of X, we can assign an orientation function ϵ for simplices, such that $\epsilon(\sigma) = \pm 1$, with 1 denoting matching orientations and -1 denoting differing orientations.

Now, define $\varphi : C_n(X) \to \mathcal{P}(X, \{1\})$ by setting its value on a single simplex σ (and extending linearly) as follows:

$$\varphi : \sigma \mapsto \begin{cases} [|\sigma|] & \text{if } \epsilon(\sigma) = 1 \text{ and } \sigma \text{ is proper} \\ -[|\sigma|] & \text{if } \epsilon(\sigma) = -1 \text{ and } \sigma \text{ is proper} \\ [|\sigma|] & \text{if } \epsilon(\sigma) = 1 \text{ and } \sigma \text{ is degenerate (not proper)} \end{cases}$$

The idea of the proof is that φ induces an isomorphism on homology:

$$\varphi_* : H_n(C_*(X)/C_*(X)^{n-1}) \xrightarrow{\sim} \mathcal{P}(X, \{1\}).$$

Note that φ vanishes of $C_n(X)^{n-1}$ by definition of the degenerate case. φ also vanishes on boundaries $\partial C_{n+1}(X)$, by a topological argument.

See [Dup01] for details; essentially there exist two decompositions of a simplex σ such that the differently-signed terms in the alternating sum $\partial\sigma$ also differ in orientation, so signs cancel and φ sends the sum to 0). φ is surjective because we can decompose any polyhedron into positively-oriented n-simplices, and their sum will map to P. Alternatively, φ is a bijection because one may construct an inverse using the simplicial approximation theorem. □

Proposition 1.4.

$$\mathcal{P}(X, G) \simeq H_0(G, \mathcal{P}(X, \{1\})).$$

Proof. This statement is a homological version of the relation (ii) $[P] = [gP]$ for all $g \in G$; in other words it is the "G" part of "G-scissors congruence". We can see this directly from the relation on $\mathcal{P}(X, G)$, and the definition of group homology.[4] Recall that group homology $H_i(M, G)$ can be defined as the ith left-derived functor of the functor that sends a G-module M to its coinvariants $M_G := M/\langle m - gm \rangle_{g \in G}$. So, $H_0(G, \mathcal{P}(X, \{1\}))$ is precisely the coinvariants of the group action of G on $\mathcal{P}(X, \{1\})$, and this agrees with our definition of $\mathcal{P}(X, G)$ as $\mathcal{P}(X, \{1\})/\langle [P] - [gP] \rangle_{g \in G}$. □

Definition 1.7 (The Steinberg module)**.** The **Steinberg module** of X is the $I(X)$-module

$$\mathrm{St}(X) := H_n(C_*(X)/C_*(X)^{n-1})$$

Remark 1.8. Above in 1.3, we had a group isomorphism $\varphi_* : \mathcal{P}(X, \{1\}) \to H_n(C_*(X)/C_*(X)^{n-1})$, but this was not quite an isomorphism of $I(X)$-modules, which is what we have for $\mathrm{St}(X)$. This is because φ introduces a twist by determinant. That is, we'll denote a twisted action of $I(X)$ on $C_n(X)$ by

$$g \cdot (a_0, \ldots, a_n) := \det(g)(ga_0, \ldots, ga_n).$$

In other words, we transform a polytope as usual by the isometry of X, and we also multiply by the determinant of g (which denotes whether it is orientation-preserving or orientation-reversing). Then one can check that the following diagram commutes:

$$\begin{tikzcd} C_n(X) \arrow[r, "\varphi"] \arrow[d, "g_{\text{twist}}\cdot"'] & \mathcal{P}(X, \{1\}) \arrow[d, "g\cdot"] \\ C_n(X) \arrow[r, "\varphi"] & \mathcal{P}(X, \{1\}) \end{tikzcd}$$

Thus $\mathcal{P}(X, \{1\}) \simeq \mathrm{St}(X)^t$ as $I(X)$-modules, where $(-)^t$ denotes the twisted action.

Proposition 1.5. *Combining our results, we have that*

$$\mathcal{P}(X, G) \simeq H_0(G, St(X)^t).$$

Upshot 1.9. The takeaway here is that we can talk about G-scissors congruence groups as the "homology of a Lie group made discrete." In other words, $\mathcal{P}(X, G)$ is the coinvariants of the Steinberg module $\mathrm{St}(X)^t$ under the action of $G = I(X)$, where *we ignore the topology of* G.

1.4 The Steinberg module is the homology of the Tits building

Remark 1.10. Above, we showed that the Steinberg module is of interest because its coinvariants are precisely the G-scissors congruence groups. Next, we will see that the Steinberg module itself has a topological origin, as the homology of a space called the Tits building.

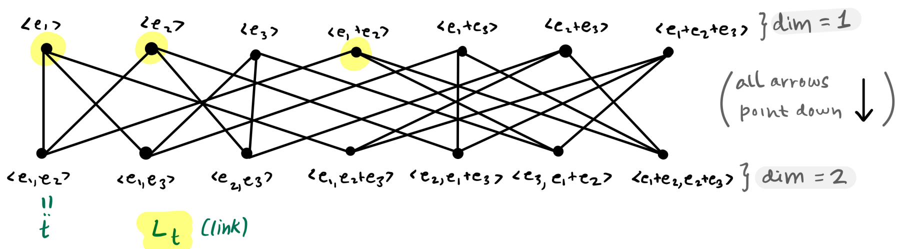
[4] In fact, we only need degree 0

Definition 1.11 (The Tits building)**.** The **Tits building** $\mathcal{T}(X)$ of a vector space X is the geometric realization of the poset of proper, nonzero subspaces of X, ordered by inclusion. Its vertices are proper nonzero subspaces of X, and a set of $(k+1)$-many vertices forms a k-simplex precisely when the vertices V_i form a flag $V_\bullet = V_0 \subset V_1 \subset \cdots \subset V_{k+1}$.

Definition 1.12 (The link of a vertex)**.** For a poset T, define the **link** of a vertex t by

$$L_t := \{s \in T \mid s < t \text{ or } s > t\}.$$

Example 1.13. The Tits building $\mathcal{T}((\mathbb{Z}/2\mathbb{Z})^3)$ consists of 14 vertices and 21 1-simplices:



Theorem 1.14 (Solomon-Tits)**.** *For $X = \mathbb{R}^n$, the Tits building $\mathcal{T}(X)$ is a wedge sum of (possibly infinitely many) $(n-2)$-spheres.*

Proof. We will use induction on $n = \dim(X)$. Note that when $n = 1, \mathcal{T}(X) = \emptyset$, because there are no proper nonzero subspaces to form the vertices. We will use $n = 2$ as a more instructive **base case**, for we have:

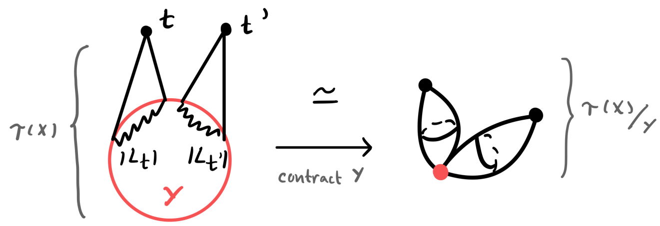
$$\{\text{Flags of proper nonzero subspaces of } X\} \leftrightarrow \{\text{Lines through 0 in } X\}$$

So vertices are precisely lines through the origin, and there are no k-simplices for $k \geq 1$. Thus we have $\mathcal{T}(X) \simeq \bigvee S^0$, a wedge sum of (infinitely many) 0-spheres.

For the **inductive step**, our hypothesis is that $\mathcal{T}(\mathbb{R}^k) \simeq \bigvee S^{k-2}$, for all $k < n$; we'll denote $X := \mathbb{R}^k$. First, fix any line $l \subset X$. Then, define

$$Y := \mathcal{T}(X) \setminus \{(n-1)\text{-dimensional subspaces not containing } l\}.$$

Note that Y is contractible; see below for a schematic drawing, or [Qui73] for a full topological proof of this claim. Collapsing Y to a point within X yields a wedge sum of coned-off links $|L_t|$, where t is a vertex, in particular an $(n-1)$-dimensional subspace that *does* contain l.



A key observation: the link $|L_t|$ consists of all subspaces of t, so

$$|L_t| \simeq |\mathcal{T}(\mathbb{R}^{n-1})| \simeq \bigvee S^{n-3},$$

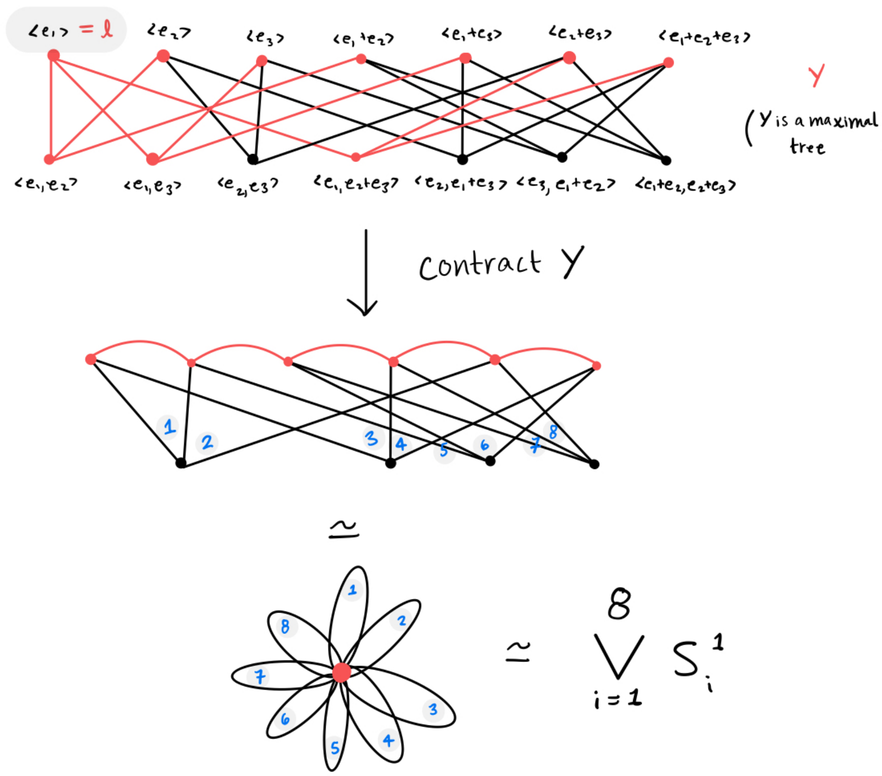
by our induction hypothesis – note that we allow $|L_t| \simeq *$ for an empty wedge sum. Thus, we have:

$$|\mathcal{T}(X)| \simeq |\mathcal{T}(X)|/Y \simeq \bigvee \sum(|L_t|) \simeq \bigvee S^{n-2}$$

□

Corollary 1.14.1. *$\mathcal{T}(X)$ has nonzero homology only in degree $n-2$; we will see that this homology is isomorphic to $St(X)$.*

Example 1.15. Let's look at an explicit example, for $X = (\mathbb{Z}/2\mathbb{Z})^3$. We will show that $\mathcal{T}((\mathbb{Z}/2\mathbb{Z})^3) \simeq S^1$. Because the field here is finite, the 3-dimensional vector space yields a Tits building with finitely many vertices; it is depicted below as a graph with 14 vertices. Fixing a vertex $l := \langle e_1 \rangle$, we can define Y, which in this case is a maximal tree in the graph $\mathcal{T}(X)$. Contracting Y leaves a wedge sum of 8 circles.



Remark 1.16. Now, it remains to see not just that $\mathcal{T}(X)$ has nonzero homology only in degree $n-2$, but also that this homology is indeed isomorphic to the Steinberg module $\mathrm{St}(X)$. The claim is the following:

Theorem 1.17.

$$H_{n-1}(\mathcal{T}(X)) \cong St(X) := H_n(C_*(X)/C_*(X)^{n-1})$$

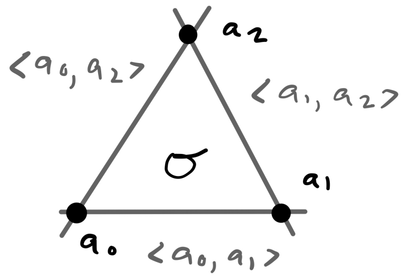
Remark 1.18. Note that homology has suddenly shifted from degree $n-2$ to $n-1$ in the statement above; this is an artifact of whether we define the Tits building to consist of simplices of flags of proper nonzero subspaces (as Quillen does, and as we've done above), or whether we "force" flags to include either $\{0\}$ or X, which induces a shift in homological degree (as Dupont does).

Proof. Omitting the technical details relating to these differing conventions, [Dup01] shows that the above isomorphism is induced by the map

$$\begin{aligned} h : C_n(X) &\to C_{n-1}(X) \\ (a_0, \dots, a_n) &\mapsto \sum_{\substack{\text{permutations} \\ \pi \text{ of } \{0,\dots,n\}}} \mathrm{sign}(\pi)(U^\pi_{n-1} \subset \dots \subset U^\pi_0), \end{aligned}$$

where the image is a sum of $(n-1)$-simplices with $U^\pi_i := \langle a_{\pi(i+1),\dots,a_{\pi(n)}} \rangle$. $\square$

Example 1.19. We'll write out an example of the map h above for $n = 2$. Consider the 2-simplex $\sigma = (a_0, a_1, a_2) \in C_2(\mathbb{R}^2)$:



We have the sum:

$$\begin{aligned} h(\sigma) = \quad & (\langle a_2 \rangle \subset \langle a_1, a_2 \rangle) && (\pi = 012) \\ & - (\langle a_2 \rangle \subset \langle a_0, a_2 \rangle) && (\pi = 102) \\ & - (\langle a_1 \rangle \subset \langle a_2, a_1 \rangle) && (\pi = 021) \\ & - (\langle a_0 \rangle \subset \langle a_1, a_0 \rangle) && (\pi = 210) \\ & + (\langle a_1 \rangle \subset \langle a_0, a_1 \rangle) && (\pi = 201) \\ & + (\langle a_0 \rangle \subset \langle a_2, a_0 \rangle) && (\pi = 120) \\ & \in C_{n-1}(\mathcal{T}(X)) \end{aligned}$$

Note that h commutes with the natural $I(X)$ actions on simplices.

3 Talk 3: Rational Structures (Oliver Wang)

Reference: [Dup01, Chapter 4].

3.1 The translational scissors congruence group

Remark 3.1. To motivate translational scissors congruence groups, let us first consider the case where $X = \mathbb{E}^n$. The isometry group of X decomposes as a semidirect product $\mathrm{Isom}(\mathbb{E}^n) \cong \mathbb{R}^n \rtimes \mathrm{O}_n$ where $\mathbb{R}^n$ acts by translation and O_n acts by linear isometries. The scissors congruence group $\mathcal{P}(X, \mathrm{Isom}(\mathbb{E}^n))$ can be expressed as iterated coinvariants as follows.

$$\begin{aligned}\mathcal{P}(X, \mathrm{Isom}(\mathbb{E}^n)) &\cong H_0(\mathrm{Isom}(\mathbb{E}^n); \mathcal{P}(X, \{1\})) \\ &\cong H_0(\mathrm{O}_n; H_0(\mathbb{R}^n; \mathcal{P}(X; \{1\})))\end{aligned}$$

As before, we regard $\mathbb{R}^n$ and O_n as discrete groups. Recall that

$$\mathcal{P}(X; \{1\}) \cong H_n\left(\frac{C_\bullet(X)}{C_\bullet(X)^{n-1}}\right)$$

where

- $C_q(X) = \mathbb{Z}[\text{(}q\text{+1)-tuples of points in } X]$,
- $C_q(X)^{n-1} = \mathbb{Z}[\text{(}q\text{+1)-tuples of points in a proper affine-linear subspace of } X]$.

Note that $C_q(X)/C_q(X)^{(n-1)} = 0$ when $q \leq n-1$. In particular, $\mathcal{P}(X; \{1\})$ is the cokernel of the map

$$\frac{C_{n+1}(X)}{C_{n+1}(X)^{n-1}} \to \frac{C_n(X)}{C_n(X)^{n-1}}.$$

For any discrete group G and any $\mathbb{Z}[G]$-module M,

$$H_0(G; M) \cong M \otimes_{\mathbb{Z}[G]} \mathbb{Z}$$

where $\mathbb{Z}$ has the trivial action. Since tensoring is left exact,

$$H_0\left(\mathbb{R}^n; \mathcal{P}(X, \{1\})\right) \cong H_0\left(\mathbb{R}^n; \frac{C_\bullet(X)}{C_\bullet(X)^{n-1}}\right) \cong H_0\left(\frac{C_\bullet(X)/\mathbb{R}^n}{C_\bullet(X)^{n-1}/\mathbb{R}^n}\right).$$

Everything in this picture except for the orthogonal group can be generalized to finite dimensional vector spaces over a characteristic zero field. This motivates the following definition:

Definition 3.2 (Translational scissors congruence groups)**.** Let $\mathbb{F}$ be a characteristic 0 field and let V be an n-dimensional $\mathbb{F}$ vector space. The **translational scissors congruence group of** V is

$$\mathcal{P}_T(V) \coloneqq H_n\left(\frac{C_\bullet(V)/V}{C_\bullet(V)^{n-1}/V}\right).$$

Question 3.3. What is the structure of $\mathcal{P}_T(V)$?

3.2 Group homology

Remark 3.4. Recall that the homology of a group with coefficients in a $\mathbb{Z}[G]$-module M is defined to be

$$H_*(G;M) := \mathrm{Tor}_*^{\mathbb{Z}[G]}(\mathbb{Z}, M) = H_*(P_\bullet \otimes_{\mathbb{Z}[G]} M)$$

where $P_\bullet$ is a projective resolution of $\mathbb{Z}$ by $\mathbb{Z}[G]$-modules. The **standard resolution** is the free resolution $C_\bullet(G)$ of $\mathbb{Z}$ by $\mathbb{Z}[G]$-modules given by $C_q(G) = \mathbb{Z}[G]^q$. The differentials are the typical alternating sums. Note that when $X = G$ this coincides with the definition of the chain complex $C_\bullet(X)$ above. It is sometimes convenient to use **bar notation** to work with this complex. This is given by the following identification:

$$g[|g_1|g_2|\cdots|g_q] := (g, gg_1, gg_1g_2, \cdots, gg_1g_2\cdots g_q).$$

Definition 3.5 (Eilenberg-Zilber map)**.** The **Eilenberg-Zilber map** is the map of chain complexes given by

$$\begin{aligned} \mathrm{EZ} : C_\bullet(G) \otimes C_\bullet(G) &\to C_\bullet(G \times G) \\ g[g_1|\cdots|g_p] \otimes g'[g'_1|\cdots|g'_q] &\mapsto \sum_{(p,q)-\text{shuffles } \sigma} \mathrm{sign}(\sigma)(g,g')[h_{\sigma(1)}|\cdots|h_{\sigma(p+q)}], \end{aligned}$$

where $h_1 = (g_1, 1), \cdots, h_p = (g_p, 1), h_{p+1} = (1, g'_1), \cdots, h_{p+q} = (1, g'_q)$.

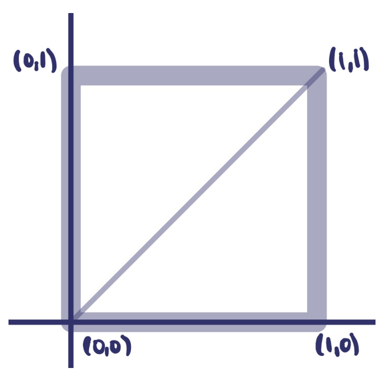
Remark 3.6. Geometrically, we may interpret the Eilenberg-Zilber map as triangulating the product of two simplices. Suppose $G = \mathbb{R}$ and consider the element $0[1] \otimes 0[1] \in C_1(\mathbb{R}) \otimes C_1(\mathbb{R})$. We can compute

$$\mathrm{EZ}(0[1] \otimes 0[1]) = (0,0)[(1,0),(0,1)] - (0,0)[(0,1)].$$

Unpacking the bar notation, we obtain

$$\mathrm{EZ}((0,1) \otimes (0,1)) = ((0,0),(1,0),(1,1)) - ((0,0),(0,1),(1,1)).$$

This should be interpreted as saying that the product $[0,1] \times [0,1]$ can be triangulated as the union of the simplices with vertices $((0,0),(1,0),(1,1))$ and $((0,0),(0,1),(1,1))$.



Remark 3.7. If G is abelian then addition is a group morphism. Composing with EZ yields the *Pontryagin product*, making $H_*(G)$ into a graded commutative ring:

$$\begin{tikzcd} H_*(G) \otimes_{\mathbb{Z}} H_*(G) \arrow[r, "\mathrm{EZ}"] \arrow[rr, bend left, "\wedge"] & H_*(G \times G) \arrow[r, "+_G"] & H_*(G) \end{tikzcd}$$

Observation. Note:

- $H_0(G) \cong \mathbb{Z}$.
- $H_1(G) \cong G$ is G is abelian.
- If G is torsionfree, then there is an isomorphism $\bigwedge^\bullet{}_{\mathbb{Z}} H_1(G) \to H_*(G)$.
- If U is a $\mathbb{Q}$-vector space, then there is an isomorphism $\bigwedge^\bullet{}_{\mathbb{Q}} U \to H_*(U)$.

The main takeaway is that for $\mathbb{Q}$-vector spaces U,

$$H_*(C_\bullet(U)/U) \cong \bigwedge{}^\bullet_{\mathbb{Q}} U.$$

3.3 The Structure of $P_T(V)$

Remark 3.8. Our goal is to compute

$$H_n\left(\frac{C_\bullet(V)/V}{C_\bullet(V)^{n-1}/V}\right).$$

For a vector space U, define the chain complex $\tilde{C}_\bullet(U)$ to be the augmented chain complex

$$\tilde{C}_\bullet(U) \coloneqq \cdots \to C_2(U) \to C_1(U) \to C_0(U) \to \mathbb{Z}$$

where $\mathbb{Z}$ is the term in degree -1. We want to consider the following double complex:

$$A_{p,q} \coloneqq \begin{cases} \tilde{C}_q(V)/V & p = -1 \\ \displaystyle\bigoplus_{(U_0 \supseteq \cdots \supseteq U_p) \in \mathcal{T}(V)_p} \tilde{C}_q(U_p)/U_p & p \geq 0. \end{cases}$$

The horizontal maps $A_{p,q} \to A_{p-1,q}$ are constructed from the face maps in the Tits building, and the vertical maps $A_{p,q} \to A_{p,q-1}$ come from the differentials in the complex $\tilde{C}_\bullet(U_p)$.

The double complex is pictured below, where we note that the $\mathbb{Z}$ in the bottom left corresponds to the index $(-1,-1)$:

$$\begin{tikzcd}
C_2(V)/V \arrow[d] & \bigoplus_{\mathcal{T}(V)_0} C_2(U_0)/U_0 \arrow[l] \arrow[d] & \bigoplus_{\mathcal{T}(V)_1} C_2(U_1)/U_1 \arrow[l] \arrow[d] & \bigoplus_{\mathcal{T}(V)_2} C_2(U_2)/U_2) \arrow[l] \arrow[d] \\
C_1(V)/V \arrow[d] & \bigoplus_{\mathcal{T}(V)_0} C_1(U_0)/U_0 \arrow[l] \arrow[d] & \bigoplus_{\mathcal{T}(V)_1} C_1(U_1)/U_1 \arrow[l] \arrow[d] & \bigoplus_{\mathcal{T}(V)_2} C_1(U_2)/U_2 \arrow[l] \arrow[d] \\
C_0(V)/V \arrow[d, "\cong"] & \bigoplus_{\mathcal{T}(V)_0} C_0(U_0)/U_0 \arrow[l] \arrow[d, "\cong"] & \bigoplus_{\mathcal{T}(V)_1} C_0(U_1)/U_1 \arrow[l] \arrow[d, "\cong"] & \bigoplus_{\mathcal{T}(V)_2} C_0(U_2)/U_2 \arrow[l] \arrow[d, "\cong"] \\
\mathbb{Z} & \bigoplus_{\mathcal{T}(V)_0} \mathbb{Z} \arrow[l] & \bigoplus_{\mathcal{T}(V)_1} \mathbb{Z} \arrow[l] & \bigoplus_{\mathcal{T}(V)_2} \mathbb{Z} \arrow[l]
\end{tikzcd}$$

If we take the horizontal homology first, we obtain the following E^1-page.

$$\begin{tikzcd}
\frac{C_2(V)}{V}/\frac{C_2(V)^{n-1}}{V} \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] \\
\frac{C_1(V)}{V}/\frac{C_1(V)^{n-1}}{V} \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] \\
0 \arrow[d] & 0 \arrow[d] & H_1(\mathcal{T}(V)) \arrow[d, "\cong"] & H_2(\mathcal{T}(V)) \arrow[d, "\cong"] \\
0 & 0 & H_1(\mathcal{T}(V)) & H_2(\mathcal{T}(V))
\end{tikzcd}$$

It follows that the differentials on E^2 vanish and there are no extension problems. We see that the homology of the double complex is

$$H_k(A_{\bullet,\bullet}) = H_{k+1}\left(\frac{C_\bullet(V)}{V}/\frac{C_\bullet(V)^{n-1}}{V}\right),$$

noting the degree shift. If we take the vertical homology of $A_{\bullet,\bullet}$ first, we end up with the following E^1 page:

$$
\begin{tikzcd}
H_2(V) & \bigoplus_{\mathcal{T}(V)_0} H_2(U_0) \arrow[l] & \bigoplus_{\mathcal{T}(V)_1} H_2(U_1) \arrow[l] & \bigoplus_{\mathcal{T}(V)_2} H_2(U_2) \arrow[l] \\
H_1(V) & \bigoplus_{\mathcal{T}(V)_0} H_1(U_0) \arrow[l] & \bigoplus_{\mathcal{T}(V)_1} H_1(U_1) \arrow[l] & \bigoplus_{\mathcal{T}(V)_2} H_1(U_2) \arrow[l] \\
0 & 0 \arrow[l] & 0 \arrow[l] & 0 \arrow[l] \\
0 & 0 \arrow[l] & 0 \arrow[l] & 0 \arrow[l]
\end{tikzcd}
$$

This yields a spectral sequence

$$E^2_{p,q} = \begin{cases} \tilde{H}_p(\mathcal{T}(V); \Lambda^q_{\mathbb{Q}}\mathfrak{g}) & q > 0 \\ 0 & q = -1, 0 \end{cases} \Rightarrow H_{p+q+1}\left(\frac{C_\bullet(V)/V}{C_\bullet(V)^{n-1}/V}\right)$$

where $\Lambda^q_{\mathbb{Q}}\mathfrak{g}$ is a certain local system associated to $(U_0, \cdots, U_p) \mapsto \bigwedge^q_{\mathbb{Q}} U_p$.

Remark 3.9. Consider the dilation operator $\mu_a : V \xrightarrow{\cdot a} V$, multiplication by an integer $a > 1$ on V. For any $U \subseteq V$, this induces a chain endomorphism $C_\bullet(U)/U \circlearrowleft$ and thus a morphism of spectral sequences. On $E^1_{p,q} = \bigoplus \bigwedge^{\bullet q} U_p$, this induces $x \mapsto xa^q$. One can show that for $r > 1$ this produces a diagram commuting with the differentials which ultimately forces $d^r = 0$.

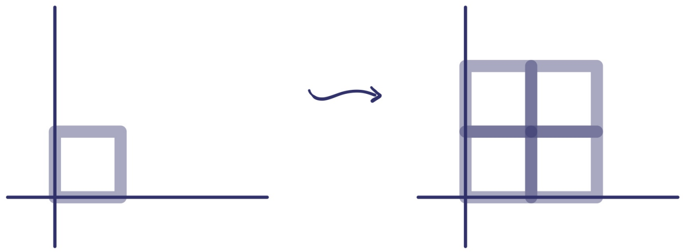
Theorem 3.10. *An analysis of the above spectral sequence yields the following conclusions:*

- $H_*\left(\frac{C_\bullet(V)/V}{C_\bullet(V)^{n-1}/V}\right)$ *is a $\mathbb{Q}$-vector space and thus so is $P_T(V)$.*
- *There is a splitting*

$$P_T(V) \cong \bigoplus_{1 \le q \le n} \tilde{H}_{n-q-1}\left(\mathcal{T}(V); \Lambda^q_{\mathbb{Q}}\mathfrak{g}\right)$$

  into a^q eigenspaces for μ_a.
- $\tilde{H}_p\left(\mathcal{T}(V); \Lambda^q_{\mathbb{Q}}\mathfrak{g}\right) = 0$ *for $p + q < n$.*

Example 3.11. What are these eigenspaces? The unit square is a a^q eigenvector for $a = q = 2$:



Remark 3.12. The groups $\tilde{H}_p(-)$ in Theorem 3.10 are similar to reduced homology in the sense that they are the homology groups of an augmented chain complex. However, $\tilde{H}_0(-)$ and $\tilde{H}_{-1}(-)$ need not vanish. Indeed, $\tilde{H}_{-1}(\mathcal{T}(V);\Lambda^n\mathfrak{g})$ consists of the a^n eigenvectors and Example 3.11 shows that these eigenvectors exist.

Theorem 3.13. *The map $\Lambda^q_{\mathbb{Q}}\mathfrak{g} \to \Lambda^q_{\mathbb{F}}\mathfrak{g}$ given by changing the field induces an isomorphism*

$$\tilde{H}_{n-q-1}\left(\mathcal{T}(V);\Lambda^q_{\mathbb{Q}}\mathfrak{g}\right) \xrightarrow{\cong} \tilde{H}_{n-q-1}\left(\mathcal{T}(V);\Lambda^q_{\mathbb{F}}\mathfrak{g}\right).$$

Using that U_{n-q} has dimension at most $q-1$, there is a containment

$$P_T(V) \subseteq \bigoplus \bigwedge\nolimits_F^q U_{n-q-1} \cong \bigoplus F,$$

*and **Hadwiger invariants***

$$P_T(V) \to \bigoplus \bigwedge\nolimits_F^q U_{n-q-1} \to \bigoplus \bigwedge\nolimits_F^q U_{n-q-1} \to F$$

Slogan 3.14. In order to understand $P_T(V)$, one should study volumes, measures, and integration on subspaces of V.

4 Talk 4: Hyperbolic Low-dimensional Theorems (Aurel Malapani)

References: [DPS88], [DS82, p. 82], and [Dup01].

4.1 Hyperbolic scissors congruence and the Bloch-Wigner sequence

Remark 4.1. Outline for the talk:

- Hyperbolic space and the half-plane model.
- The extended hyperbolic place, $\mathcal{P}(\overline{\mathbb{H}})$, and $\mathcal{P}(\partial\mathbb{H})$.
- The Bloch-Wigner spectral sequence.

4.2 Hyperbolic space

Definition 4.2 (The half-space model)**.** The half-plane model of hyperbolic space is defined as

$$\mathbb{H}^n := \left\{ [x_1, \cdots, x_n] \in \mathbb{R}_n \;\middle|\; x_n > 0 \right\},$$

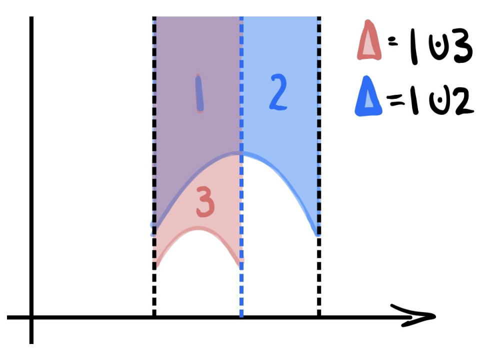
which just excludes one axis. Geodesics are lines or circles with endpoints on $\mathbb{R}^{n-1}$, and $\partial\mathbb{H}^n = \mathbb{R}^n \cup \{\infty\}$. We define $\overline{\mathbb{H}}^n := \mathbb{H}^n \cup \partial\mathbb{H}^n$.

Definition 4.3 (Hyperbolic scissors congruence groups)**.** $\mathcal{P}(\mathbb{H}^n)$ is the free abelian group on symbols $[P]$ where $P \hookrightarrow \mathbb{H}^n$ is a polytope in $\mathbb{H}^n$ with relations

- $[P] = [P'] + [P'']$ when $P' \uplus P'' = P$, and
- $[P] = [gP]$ for all $g \in G := \mathrm{Isom}(\mathbb{H}^n)$.

Theorem 4.4 (Zylev)**.** *If $[P] = [Q]$ in $\mathcal{P}(\mathbb{H}^n)$ then P, Q are scissors congruent.*

Example 4.5. Recall from Example 0.15 that this does not hold for $\overline{\mathbb{H}}^n$, as exemplified by the following picture:



Theorem 4.6 ([DS82], Theorem 2)**.** *The inclusion $\mathbb{H}^n \hookrightarrow \overline{\mathbb{H}}^n$ induces an isomorphism $\mathcal{P}(\mathbb{H}^n) \xrightarrow{\cong} \mathcal{P}(\overline{\mathbb{H}}^n)$.*

Proof sketch. Write

$$\mathcal{P}(X) \cong H_0(G; \mathrm{St}(X)^t) \qquad \text{where } \mathrm{St}(X) = \tilde{H}_{n-1}(\tau(X); \mathbb{Z}).$$

For any ideal point $p \in \partial\mathbb{H}^n$, let $\tau(\overline{\mathbb{H}}^n, p)$ be the Tits complex of flags containing p, and note that $\mathrm{St}(\overline{\mathbb{H}}^n) = \tilde{H}_{n-2}$.

Lemma 4.7. *There is a short exact sequence of G-representations*

$$St(\mathbb{H}^n) \hookrightarrow St(\overline{\mathbb{H}}^n) \twoheadrightarrow \coprod_{p \in \mathbb{H}^n} St(\overline{\mathbb{H}}^n, p).$$

The proof follows from considering the long exact sequence for the pair $T(\bar{\mathbb{H}}), T(\mathbb{H}))$. Taking the long exact sequence in homology, it suffices to show

$$H_k\left(G, \coprod_{p \in \partial\mathbb{H}^n} \mathrm{St}(\overline{\mathbb{H}}^n, p)^t\right) = 0, \qquad k = 0, 1.$$

It turns out that this is zero for all k. By Shapiro's lemma, one can write

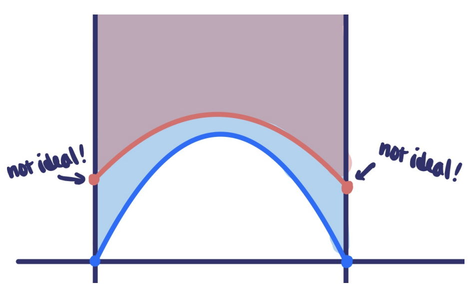
$$H_k\left(G, \coprod_{p \in \partial\mathbb{H}^n} \mathrm{St}(\overline{\mathbb{H}}^n, p)^t\right) \xrightarrow{\cong} H_*(\mathrm{Sim}(n-1), \mathrm{St}(\mathbb{R}^{n-1})).$$

One can conclude using the Hochschild-Serre spectral sequence for

$$1 \to T(n-1) \to \mathrm{Sim}(n-1) \to \mathrm{Sim}_0(n-1) \to 1,$$

where Sim_0 denotes similarities fixing the origin. □

Remark 4.8. We would like to define $\mathcal{P}(\partial\mathbb{H}^n)$, but it is not clear how to do this. For example, we can take a triangle whose vertices live in $\partial\mathbb{H}^n$, act by translation, and get a triangle whose vertices are not ideal points:



So we copy the homological definitions, defining $G := \mathrm{Isom}(\mathbb{H}_n)|_{\partial\mathbb{H}_n}$ and

$$\mathcal{P}(\partial\mathbb{H}^n) := H_0(G; \mathrm{St}(\partial\mathbb{H}^n)^t), \qquad C_\bullet(\partial\mathbb{H}^n)/C_\bullet(\partial\mathbb{H}^n)^{n-1}.$$

Explicitly, this is the free Abelian group on $(n+1)$-tuples on exact sequences of ideal points $\mathbf{a} := (a_0, \cdots, a_n)$ with the following relations:

- $\mathbf{a} = 0$ iff it lies in a hyperplane
- $\sum(-1)^i(a_0, \cdots, \widehat{a}_i, \cdots a_n) = 0$.
- $(ga_0, \cdots, ga_n) = \det(g)\mathbf{a}$.

Proposition 4.1 (DS82 3.7)**.** *There is an exact sequence*

$$H_1(\mathrm{O}_n, St(\mathbb{S}^{n-1})^t) \to \mathcal{P}(\partial\mathbb{H}^n) \to \mathcal{P}(\partial\mathbb{H}^n) \twoheadrightarrow H_0(\mathrm{O}_n; St(\mathbb{S}^{n-1})^t).$$

Proof. Similar to the previous theorem, but use a different short exact sequence:

$$\mathrm{St}(\partial\mathbb{H}^n) \hookrightarrow \mathrm{St}(\overline{\mathbb{H}}^n) \twoheadrightarrow \coprod_{p\in\mathbb{H}^n} \mathrm{St}(\mathbb{H}^n, p),$$

noting that the last term is now over only interior points.

□

Example 4.9. It turns out that $\mathcal{P}(\mathbb{H}^2) \xrightarrow{\cong} \mathcal{P}(\overline{\mathbb{H}}^2)$. In this case, $\partial\mathbb{H}^2 = \mathbb{R}\cup\{\infty\}$, $G = \mathrm{PSL}_2(\mathbb{R}) \rtimes C_2$, and $G \curvearrowright \mathbb{H}^2$ by

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} x = \frac{ax+b}{cx+d}$$

where C_2 acts by -1. In this case, $\mathcal{P}(\partial\mathbb{H}^2) \cong \mathbb{Z}$, and we get a short exact sequence

$$\mathbb{Z} \hookrightarrow \mathcal{P}(\overline{\mathbb{H}}^2) \twoheadrightarrow \mathbb{R}/\mathcal{P}\mathbb{Z}$$

and $\mathcal{P}(\overline{\mathbb{H}}^2) \cong \mathcal{P}(\mathbb{H}^2) \cong \mathbb{R}$, where the second isomorphism is via area.

4.3 Bloch-Wigner and $\mathcal{P}(\mathbb{H}^3)$

Remark 4.10. Using the upper half-space model for $\mathbb{H}^3$ yields $G \cong \mathrm{PSL}_2(\mathbb{C}) \rtimes C_2$, identifying $\partial\mathbb{H}^3 \cong \mathbb{CP}^1$. Then C_2 acts by $z \mapsto -\bar{z}$ and $\mathrm{PSL}_2(\mathbb{C})$ acts by

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} z = \frac{az+b}{cz+d}.$$

Let $\mathbb{P}_{\mathbb{C}}$ denote the coinvariants of the action on $\partial\mathbb{H}^3$. We get a new relation:

- $\mathbf{a} = 0$ iff it lies in a hyperplane
- $\sum(-1)^i(a_0, \cdots, \widehat{a}_i, \cdots a_n) = 0$.
- $(ga_0, \cdots, ga_n) = \mathbf{a}$.

For any $\mathbf{a} \in \mathbb{H}^3$, there is a $g \in G$ such that

$$\mathbf{a} = g(\infty, 0, 1, z), \qquad z := \frac{a_0 - a_2}{a_0 - a_3} \cdot \frac{a_1 - a_3}{a_1 - a_2} \in \mathbb{C} \setminus \{0, 1\}.$$

Theorem 4.11 (Bloch-Wigner)**.** *There is an exact sequence*

$$\mathbb{Q}/\mathbb{Z} \overset{\alpha}{\hookrightarrow} H_3(\mathrm{SL}_2(\mathbb{C})) \xrightarrow{F} P_{\mathbb{C}} \xrightarrow{\gamma} \bigwedge^2 \mathbb{C}^\times \twoheadrightarrow \mathsf{K}_2(\mathbb{C}) \xrightarrow{\cong} H_2(\mathrm{SL}_2(\mathbb{C}); \mathbb{Z}).$$

Remark 4.12. Some things to note:

- Elements in $\mathsf{K}_2(\mathbb{C})$ are written in brackets $\{a, b\}$ due to the isomorphism $\mathsf{K}_M(\mathbb{C}) \xrightarrow{\cong} \mathsf{K}_2(\mathbb{C})$.
- $\delta(a \wedge b) = \{a, b\}$.
- α is induced by

$$\begin{aligned} \mathbb{Q}/\mathbb{Z} &\to \mathrm{SL}_2(\mathbb{C}) \\ z &\mapsto \begin{bmatrix} z & 0 \\ 0 & z^{-1} \end{bmatrix}. \end{aligned}$$

- $\beta[g_1|g_2|g_3] = (\infty, g_1\infty, g_1g_2\infty, g_1g_2g_3\infty)$
- $\gamma\{z\} = z \wedge (1 - z)$.[5]

Theorem 4.13 (DS82 5.1)**.** *We can define $\mathcal{P}_{\mathbb{F}}$ in a similar way for any field $\mathbb{F}$. When $\mathbb{F}$ is algebraically closed, $P_{\mathbb{F}}$ is a divisible group.*

Theorem 4.14. *For $\mathbb{H}^3$,*

$$\mathcal{P}(\partial\mathbb{H}^3) \xrightarrow{\cong} \mathcal{P}(\mathbb{H}^3) \xrightarrow{\cong} \mathcal{P}(\overline{\mathbb{H}}^3),$$

so the Bloch-Wigner sequence deeply relates to all of these groups.

Remark 4.15. The Bloch-Wigner dilogarithm can be realized as a map out of the Bloch group $P_{\mathbb{C}}$.

Remark 4.16. Are there any links to hyperbolic 3-manifolds of the form $X = \mathbb{H}^3/\Gamma$? Jonathan's answer: ask Daniil Rudenko!

[5] Proving this involves a hypercohomology spectral sequence, and γ is an awful transgression map.

5 Talk 5: The Polytope Algebra (Kyle Huang)

References: [McM89], [Goo14].

Remark 5.1 (Motivation)**.** The polytope algebra generalizes the polytope groups of previous talks, but with extra structure:

- Keeps track of lower dimensional data (what Johnathan calls the cutting "sawdust") .
- Many nice results hold and are proven analogously.
- Interesting/useful in many fields:
  - Discrete geometry: one direction of the g-theorem,
  - Tropical geometry,
  - Toric geometry: Chow groups of toric varieties,
  - K-theory: what we're talking about today!

Also has a more topological flavor, and is used to prove generalized and refined Dehn invariants (Goodwillie).

Definition 5.2 (The polytope algebra)**.** Let $\mathcal{P}$ denote the free abelian group of polytopes on $\mathbb{R}^d$. Consider the relations

1. $[P \cup Q] + [P \cap Q] = [P] + [Q]$, and
2. $[P + t] = [P]$ for any translation $t \in \mathbb{R}^d$.

The **polytope algebra** Π is the quotient of $\mathcal{P}$ by the above relations.

Example 5.3. An example of an element $x \in \Pi$ is

$$x = 3[\Delta] + 2[\Box] - [\text{line segment}].$$

Elements of Π can also represent more general geometric objects than the polytope group, for example manifolds without boundary by subtracting the boundary.

Proposition 5.1. *Due to Sallee, item 2 can be replaced by*

$$[P] + [P \cap H] = [P \cap H^-] + [P \cap H^+],$$

where H is a hyperplane and H^-, H^+ are the half-spaces on either side of H.

Proof idea. Consider $P \cap Q$ and induct on the number of facets. $\Box$

Question 5.4. What is the multiplicative/algebra structure on the polytope algebra?

Definition 5.5 (Multiplicative structure on the polytope algebra)**.** For $[P], [Q] \in \Pi$, define their product to be

$$[P] \cdot [Q] = [P + Q] \coloneqq \{a + b \mid a \in P, b \in Q\},$$

an operation called the **Minkowski sum**. Proving the distributivity relation $x(y + z) = xy + xz$ uses the scissors congruence relation in the definition of Π.

Example 5.6. One should be careful about extending Minkowski sums to arbitrary elements of Π. For example if we have two points in $\mathbb{R}^1$ like $0, 1$ and a line segment $[0, 2]$ then their Minkowski sum is **not** $[0, 3]$ – instead it is $[0, 2] + [1, 3]$!

Remark 5.7. Note that this is a commutative product with unit a singleton point, and thus we have the structure of a commutative unital ring. What other structures can we discover?

5.1 Endomorphisms

Remark 5.8. If $\varphi : \mathbb{R}^d \to \mathbb{R}^d$ is an affine map, then there is an induced algebra endomorphism

$$\begin{aligned} \hat{\varphi} : \Pi &\to \Pi \\ [P] &\mapsto [\varphi(P)] \end{aligned}$$

We defined ϕ on representatives, so we simply need to check that it behaves nicely with respect to the equivalence relations and also the product structure of Minkowski sums.

Corollary 5.8.1. *For every $\lambda \in \mathbb{R}_{\geq 0}$, there is a dilation map*

$$\cdot\lambda : \mathbb{R}^d \to \mathbb{R}^d$$

which yields an algebra endomorphism $\cdot\lambda : \Pi \to \Pi$.

Remark 5.9. The important point is that we start with something which is affine. There is a grading on the polytope algebra which makes it almost graded-commutative.

5.2 Filtration and rational structure

Remark 5.10. First define Π_0 to be sums of points in $\mathbb{R}^d$, and note that $\Pi_0 \simeq \mathbb{Z}$ since all points in $\mathbb{R}^d$ are equal via translation.

Lemma 5.11. *Let Z_1 be the ideal generated by $[P] - 1$ for $P \neq \varnothing$. Then Π has a decomposition $\Pi = \Pi_0 \oplus Z_1$. Furthermore, Z_1 is the kernel of dilation by zero $\Delta(0) : \Pi \to \Pi$.*

Proof. McMullen uses a claim[6] here that valuations (morphisms out of Π to an abelian group) are invariant on simplices of the same dimension, via the claim in section 5.1 about endomorphisms, since there exists an affine isomorphism bringing a k-simplex to any other k-simplex. The outline is as follows:

[6] To me, dubious.

1. Prove for simplices, i.e. that $\Delta(0)([\Delta]-1)=0$.
2. Extend to polytopes since every polytope is triangulable, hence a sum (with $\mathbb{Z}$-coefficients) of simplices of various dimensions.

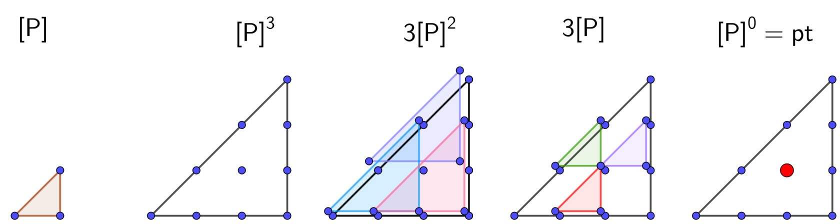
The claim is used to show that that $\Delta(0)$ is equal on same-dimensional simplices. So we can apply a hyperplane cut to any k-simplex to reduce the dimension, then use induction. □

Lemma 5.12.

$$([P]-1)^r = 0 \qquad \textit{for } r > d$$

Remark 5.13. They prove something much more general about canonical simplicial dissections, but I believe the lemma can be reached much more quickly via a direct geometric argument inspired by an argument in *Ehrhart theory* using inclusion-exclusion. The alternate proof is as illustrated in the following example:

Example 5.14. Let $d=2, r=3$. First note that in general $P^k = kP$, i.e. that k-fold Minkowski sum is equal to its k-th dilate. Then we can tile $(d+1)P$ with smaller dilates of P via inclusion-exclusion, as illustrated in the following picture when $d=2$ and P is a unimodular right triangle:



Definition 5.15. Let Z_r be the ideal in Π generated by elements of the form $([P]-1)^r$.

Remark 5.16. Observe that $Z_r \subseteq Z_{r_1}$. This gives us a filtration

$$\Pi : Z_0 \supset Z_1 \supset Z_2 \cdots \supset Z_{d+1} = 0$$

where the last term is zero and the filtration is finite by a separate lemma.

5.3 Rational structure

Remark 5.17. The following lemma will be useful in proving the existence of a rational structure on Π. Loosely speaking, it lets us travel down the fibration and use induction:

Lemma 5.18. *Let $x \in Z_r$, then $\Delta(n)x - n^r x \in Z_{r+1}$.*

Proof. Suppose $x = ([P] - 1)^r$. Given the previous lemma 5.12 note that

$$\Delta(n)([P] - 1) = \sum_{k=1}^{n} \binom{n}{k} ([P] - 1)^k.$$

Taking rth powers of both sides, we see that

$$\Delta(n)([P] - 1)^r = (\Delta(n)([P] - 1))^r = \sum_{k=1}^{n} \binom{n}{k} ([P] - 1)^k,$$

where the first equality is because $\Delta(n)$ is an algebra morphism. On the right hand side, note that all but one term is clearly in Z_{r+1}, so the only case we need to be careful about is $k = 1$. However,

$$\left(\binom{n}{k}([P] - 1)\right)^r = n^r x,$$

which concludes the proof. □

Lemma 5.19. *Z_1 is torsion-free.*

Proof. Let $x \in Z_1$ such that $nx = 0$ where $n \in \mathbb{Z}_{\neq 0}$. It suffices to show that $x = 0$.

$$\Delta(n)x - n^{r-1}nx \in Z_{r+1} \implies \Delta(n)x \in Z_{r+1}.$$

Since $x = \Delta(n^{-1})\Delta(n)x \in Z_{r+1}$, something we can show that x is in Z_m for $m \gg 0$, which implies that $x = 0$ because the filtration is finite. □

Lemma 5.20. *Z_1 is divisible.*

Proof. Let $x \in Z_1$, $m \geq 2$ an integer. We will induct. Our base case is $x \in Z_d$. Then

$$x = \Delta(m)\Delta(m^{-1})x = mm^{d-1}\Delta(m^{-1})$$

and hence we have found $m^{-1}x$. □

6 Talk 6: Assemblers (Brandon Shapiro)

References: [Zak16] and [Zak].

6.1 Sieves and sites

Remark 6.1. In Talk 0, Inna talked about K-theory as a way to encode *finite decompositions*. We talked about the three-term relation $B = A + C$ which shows up in multiple settings: Classical K-theory focuses mostly on R-modules,

R-modules	Short exact sequences $A \to B \to C$
Finite sets	Inclusions $A \to B \leftarrow C = B \setminus A$
Varieties	$A \to B \leftarrow C = B \setminus A$[7]
Polytopes	$A \to B \leftarrow C = B \setminus A$

but we want to focus more on the other rows which are more combinatorial. Assemblers give us a formalism within which we can do this.

Assumption: Throughout, we will assume $\mathcal{C}$ is a category with pullbacks.[8]

Note that it is not enough to just have pair of inclusions – we need them to be "complementary" in the sense that they "cover" the middle object. We need a way to express what this means in the language of category theory.

Definition 6.2 (Sieves)**.** A **sieve** on $C \in \mathcal{C}$ is a full subcategory of $\mathcal{C}_{/C}$ closed under precomposition.

$$\begin{tikzcd}
\cdots \arrow[r] & A \arrow[dr] & \\
\cdots \arrow[ur] \arrow[dr] & & C \\
\cdots \arrow[r] & B \arrow[ur] &
\end{tikzcd}$$

Any collection of morphisms $\{A_i \to A\}_{i \in I}$ generates a sieve on $A \in \mathrm{Ob}\,\mathcal{C}$.

Definition 6.3 (Pullbacks of sieves)**.** For $f\colon B \to C$ and some sieve S on C, define f^*S by pullback

$$\begin{tikzcd}
\vdots & & \vdots & \vdots & & \vdots \\
f^*C_1 \arrow[dr] \arrow[rrr, bend left, "f"] & & f^*C_2 \arrow[dl] \arrow[rrr, bend left, "f"] & C_1 \arrow[dr] & & C_2 \arrow[dl] \\
& B \arrow[rrr, bend left, "f"] & & & C &
\end{tikzcd}$$

[8] The pullback assumption is not necessary but makes things easier to think about.

Definition 6.4 (Grothendieck topologies)**.** A **Grothendieck topology** on $\mathcal{C}$ consists of collections $J(C)$ of sieves (called "covering sieves") for each object C such that

- For all $S \in J(C)$ and $f \colon B \to C$, $f^*S \in J(B)$,
- For all $S \in J(C)$ and T a sieve on C, if $f^*T \in J(C_i)$ for all $C_i \xrightarrow{f} C$ in S, then $T \in J(C)$,
- $\mathcal{C}_{/C} \in J(C)$, where $\mathcal{C}_{/C}$ is generated by $\mathrm{id}_C \colon C \to C$.

Definition 6.5. A **closed**[9] **assembler** is a category C with a Grothendieck topology such that

- $\mathcal{C}$ has an initial object $\emptyset$ and the empty sieve covers it, and
- All morphisms in $\mathcal{C}$ are monic.

Example 6.6. (Polytopes) Let $\mathcal{G}_n$ be the assembler with

- Objects: n-dimensional closed polytopes in $\mathbb{R}^\infty$ (not necessarily connected and also including $\emptyset$),
- Morphisms: inclusions-after-isometry (includes choice of isometry)
- Grothendieck topology: $\{A_i \to A\}_{i\in I}$ generate a cover when $A = \cup_i A_i$.

Let $\mathcal{O}$ be the assembler of all polytopes (not necessarily closed) in any dimension, with similar morphisms and topology.

Note that there is no condition for $A = \cup_i A_i$ in this definition that the cover is disjoint or intersections have measure 0. The next definition introduces this idea.

Definition 6.7 (Disjoint covers)**.** A cover is **disjoint** if it generated by $\{A_i \to A\}_{i\in I}$ such that the square

$$\begin{array}{ccc} \emptyset & \longrightarrow & A_i \\ \big\downarrow & & \big\downarrow \\ A_j & \longrightarrow & A \end{array}$$

is a pullback for all $i \neq j$.

Example 6.8. In $\mathcal{O}$ this manifests as disjointness, and in $\mathcal{G}_n$ this means the intersection is not n-dimensional and thus has measure 0.

[9]This means $\mathcal{C}$ has pullbacks. Without the pullback assumption, there is an extra axiom in the definition of assembler.

6.2 K-theory of assemblers and scissors congruence

Remark 6.9. To get a K-theory of assemblers, we want a space (really, a spectrum) $\mathsf{K}(\mathcal{C})$ for an assembler $\mathcal{C}$, and then we can define the K-groups as $\mathsf{K}_i(\mathcal{C}) \coloneqq \pi_i \mathsf{K}(\mathcal{C})$. Normally in K-theory, we need to be able to not only decompose things but also combine them, which motivates the next definition.

Definition 6.10 (Formal sums)**.** A **formal sum** X over $\mathcal{C}$ consists of a finite set I and for all $i \in I$ an object A_i of $\mathcal{C}$ which is non-initial. Write $X = (I, \{A_i\})_{i\in I})$. There are two different kinds of morphisms between formal sums $(I, \{A_i\})$ and $(J, \{B_j\})$:

- (monic) move: $(I, \{A_i\}_{i\in I}) \rightarrowtail (J, \{B_j\}_{j\in J})$ is an inclusion $f\colon I \hookrightarrow J$ along with $A_i \cong B_{f(i)}$ for all $i \in I$,
- weak equivalence: $(I, \{A_i\}_{i\in I}) \xrightarrow{\sim} (J, \{B_j\}_{j\in J})$ is a surjection $f\colon I \twoheadrightarrow J$ along with a disjoint cover $\{A_i \to B_j\}_{i\in f^{-1}(j)}$ for all $j \in J$.

When J is a singleton, a weak equivalence is given by a disjoint cover $\{A_i \to B\}_{i\in I}$.

Definition 6.11 (Scissors-congruent objects)**.** Objects A and B in C are **scissors congruent** if there exist weak equivalences

$$(I, \{A_i\}_{i\in I}) \xrightarrow{\sim} (*, A),$$
$$(I, \{B_i\}_{i\in I}) \xrightarrow{\sim} (*, B)$$

such that $A_i \cong B_i$ for all $i \in I$.

Remark 6.12. This captures the idea that A and B can be decomposed into "the same" (isomorphic) collection of smaller pieces.

Definition 6.13 (K-theory of an assembler)**.** If $\mathcal{C}$ is a closed assembler, define

$$\mathsf{K}(\mathcal{C}) := \Omega\,|wS_\bullet\mathcal{C}|\,.$$

Remark 6.14. We will see that $wS_\bullet\mathcal{C}$ is a simplicial category. Then $|wS_\bullet\mathcal{C}|$ is the realization of the diagonal simplicial set $[n] \mapsto N_n wS_n\mathcal{C}$, and Ω denotes taking the loop space.

The space $\mathsf{K}(\mathcal{C})$ is therefore constructed as an example of Waldhausen K-theory, making it automatically an infinite loop space (aka a loop spectrum). This definition of $\mathsf{K}(\mathcal{C})$ is presented in [Zak], where Inna shows it to be equivalent to the original definition given in [Zak16].

Remark 6.15. So what is $wS_n\mathcal{C}$?

- Objects: n-tuple of spans of formal sums which we think of as the "cofibrations":
$$X_0 \xleftarrow{\sim} \bullet \rightarrowtail X_1 \xleftarrow{\sim} \bullet \rightarrowtail \dots \xleftarrow{\sim} \bullet \rightarrowtail X_n$$
Basically, we want to think of the monics $\rightarrowtail$ as cofibrations but to do so we need to invert weak equivalences.

- Morphisms: natural transformations of these diagrams whose components are weak equivalences:

$$\begin{tikzcd}
X_0 \arrow[d, "\sim"] & \bullet \arrow[l, "\sim"'] \arrow[r, tail] \arrow[d, "\sim"] & X_1 \arrow[d, "\sim"] & \bullet \arrow[l, "\sim"'] \arrow[r, tail] \arrow[d, "\sim"] & \dots & \bullet \arrow[l, "\sim"'] \arrow[r, tail] \arrow[d, "\sim"] & X_n \arrow[d, "\sim"] \\
X_0' & \bullet \arrow[l, "\sim"'] \arrow[r, tail] & X_1' & \bullet \arrow[l, "\sim"'] \arrow[r, tail] & \dots & \bullet \arrow[l, "\sim"'] \arrow[r, tail] & X_n'
\end{tikzcd}$$

  We require the squares of weak equivalences to be pullbacks and that all the squares commute (in the category whose morphisms $(I,(A_i)) \to (J,(B_j))$ are given by functions $f\colon I \to J$ and disjoint covers $\{A_i \to B_j\}_i \in f^{-1}(j)$ for all $j \in J$).

In terms of the scissors congruence story, each $\bullet$ contains the pieces we have cut an object up into, the weak equivalences encode this cutting, and the monics reassemble the pieces into a part of another object. This defines a K-theory and fits into other K-theory frameworks, e.g. Waldhausen K-theory.

Proposition 6.1. $\mathsf{K}_0(\mathcal{C}) \cong \mathbb{Z}[\mathrm{Ob}\mathcal{C}]/\sim$ *where* $[A] = \sum_i [A_i]$ *for any disjoint cover* $\{A_i \to A\}_i$.

Remark 6.16. Note that weakly equivalent objects are identified. This recovers what we want K-theory to be: disjoint covers are treated as sums.

Example 6.17. If $\mathcal{C} = \emptyset \to *$ where $* \to *$ is the only cover, then $\mathsf{K}(\mathcal{C}) \simeq \mathcal{S}$, the sphere spectrum. The formal sums look like finite sets, so we recover the K-theory of finite sets.

Example 6.18. $\mathsf{K}_0(\mathcal{G}_n)$ recovers the scissors congruence group of n-dimensional polytopes and $\mathsf{K}_0(\mathcal{O})$ is isomorphic to the scissors congruence group of all polytopes.

Remark 6.19. We can do generalized operations on assemblers to get a cofiber sequence

$$\mathsf{K}(\mathcal{D}) \to \mathsf{K}(\mathcal{C}) \to \mathsf{K}(\mathcal{C} \setminus \mathcal{D})$$

for suitably defined assemblers $\mathcal{C}, \mathcal{D}$ plus some conditions. Specifically, we ask that $\mathcal{D}$ is subassembler of $\mathcal{C}$, and every morphism $D \to C$ in $\mathcal{C}$ (for $D \in \mathrm{Ob}\,\mathcal{D}$ and $C \in \mathrm{Ob}\,\mathcal{C}$) belongs to a disjoint cover of C. It turns out $C \setminus D$ is also an assembler, essentially the complement of D in C with the initial object added back in.

We can apply this to $\mathcal{O}$, which has a filtration given by

$$\mathcal{O}_0 \hookrightarrow \mathcal{O}_1 \hookrightarrow \cdots \hookrightarrow \mathcal{O}$$

where $\mathcal{O}_n$ contains polytopes of dimension $\leq n$. Applying the cofiber sequence above, we get cofiber sequences

$$\mathsf{K}(\mathcal{O}_{n-1}) \to \mathsf{K}(\mathcal{O}_n) \to \mathsf{K}(\mathcal{G}_n),$$

which can help us compute scissors congruence groups of polytopes.

Remark 6.20. "What have we discussed might be called "combinatorial" K-theory, as opposed to algebraic K-theory. For instance, we cannot do K-theory of rings this way, since there's no good Grothendieck topology on R-modules.

What have we discussed might be called "combinatorial" K-theory, as opposed to algebraic K-theory. For instance, we cannot do K-theory of rings this way, since there's no good Grothendieck topology on R-modules. However, most of the homotopical tools for algebraic K-theory (e.g. cylinder functors) will not work here, and familiar statements like "every morphism is a cofibration up to weak equivalence" are no longer true.

7 Talk 7: Grothendieck Ring of Varieties (Michael Montoro)

Reference: [NS11], [Bor15], and [Nic09].

7.1 Definitions

Let S be a fixed Noetherian scheme. Denote Iso_S to be the isomorphism classes of finite type separated S-schemes.

Definition 7.1 (Grothendieck ring of varieties)**.** The **Grothendieck ring of varieties** $\mathsf{K}_0(\mathsf{Var}_S) = \langle \mathrm{Iso}_S \rangle / \sim$ is the free abelian group generated by isomorphism classes of S-varieties, modulo the relation $[X] = [Y] + [X \setminus Y]$, where Y is a closed subscheme of X, a representative of a class in Iso_S. The ring structure on $\mathsf{K}_0(\mathsf{Var}_S)$ is given by $[X] \cdot [Y] = [X \underset{S}{\times} Y]$.

Notation. If $S = \operatorname{Spec} k$ for a field k, we will denote $\mathsf{K}_0(\mathsf{Var}_k) \coloneqq \mathsf{K}_0(\mathsf{Var}_S)$. The **Lefschetz motive** is $\mathbb{L} \coloneqq [\mathbb{A}^1_S]$.

Remark 7.2. $\mathsf{K}_0(\mathsf{Var}_S)$ has the following properties.

1. $[\varnothing] = 0$ and $[S] = 1$.
2. $[\mathbb{P}^n_S] = 1 + \mathbb{L} + \mathbb{L}^2 + \cdots + \mathbb{L}^n$, which can be shown inductively.
3. $[X_{\mathrm{red}}] = [X]$, so we can assume that the varieties are reduced.
4. Given a map $f \colon T \to S$, we have induced maps

$$\mathsf{K}_0(\mathsf{Var}_S) \xrightarrow{f^*} \mathsf{K}_0(\mathsf{Var}_T)$$
$$\mathsf{K}_0(\mathsf{Var}_S) \xleftarrow{f_!} \mathsf{K}_0(\mathsf{Var}_T)$$

given by

$$f^*([X]) = [X \underset{S}{\times} T],$$
$$f_!([Y]) = [Y|_S].$$

Definition 7.3 (Piecewise isomorphic varieties)**.** We say that two S-varieties X and Y are **piecewise isomorphic** if there exist locally closed subvarieties $\{X_i\}_{i \in I}$ of X and $\{Y_j\}_{j \in J}$of Y such that $X = \bigcup_i X_i$, $Y = \bigcup_j Y_j$, and there is a bijection $\sigma \colon I \to J$ such that $X_i \cong Y_{\sigma(i)}$.

Notation. Denote
$$\mathsf{K}_0(\mathsf{Var}_S)[\mathbb{L}^{-1}] = \mathcal{M}_S.$$

Why should we care about this? $\mathsf{K}_0(\mathsf{Var}_S)$ is "the universal home for additive invariants": Suppose we have a map $f\colon \mathrm{Iso}_S \to A$ for a commutative ring A. If this map takes disjoint union to addition, and fiber products to multiplication, then F factors through $\mathsf{K}_0(\mathsf{Var}_S)$.

Example 7.4. The following are all additive invariants:

1. For $k = \mathbb{F}_q$, the point counting map, which counts the number of $\mathbb{F}_q$-rational points on a variety,
2. For $k = \mathbb{C}$, the Hodge characteristic,
3. Euler characteristics of finite type schemes over algebraically closed fields, or ℓ-adic Galois representations.

7.2 Zeta functions

The **Hasse–Weil zeta function** is the generating function

$$Z(X,t) = \sum_{n\geq 0} |\mathrm{Div}_n(X)|\, t^n = \prod_{x\in X_{\mathrm{cl}}} (1 - t^{\deg(x)})^{-1},$$

where $\mathrm{Div}_n(X)$ is the set of effective 0-divisors of degree n (formal sums of n closed points up to multiplicity), and X_{cl} denotes the set of closed points in X. This is defined on equivalence classes of varieties, so we have

$$Z([X],t) = \sum [\mathrm{Div}_n(X)]t^n \in \mathsf{K}_0(\mathsf{Var}_k)[[t]].,$$

where we view $\mathrm{Div}_n(X) = \bigwedge^n X$ as an S-scheme. We'd like to interpret this as some kind of motivic integral, mimicking the fact that for p-adic integration you want to rewrite the zeta function as an integral.

Definition 7.5 (Jet bundles)**.** Let X be a variety over a field k, with fixed pure dimension d. The $n^{\mathbf{th}}$ **jet bundle** of X is

$$\mathcal{L}_n(X) = \mathrm{Hom}(\mathrm{Spec}(k[t]/t^{n+1}), X).$$

This is a variety since $\mathcal{L}_n$ is a representable functor. Note that $\mathcal{L}_0(X) = X$, and $\mathcal{L}_1(X)$ is the tangent bundle of X, i.e. the $k[t]/(t^2)$-points of X.

Consider the inverse limit

$$\mathcal{L}_\infty(X) = \varprojlim \mathcal{L}_n(X) \cong \mathrm{Hom}(\mathrm{Spec}\ k[[t]], X).$$

This is also a representable functor. We call it the **infinite jet bundle** (or **infinite jet scheme**) of X.

To define an integral over $\mathcal{L}_\infty(X)$, we need to define the measure. Assume X is smooth. We have maps

$$\mathcal{L}_\infty(X) \xrightarrow{\pi_n} \mathcal{L}_n(X).$$

Let $C = \pi_n^{-1}(C_n)$ be the preimage of a constructible set $C_n \subset \mathcal{L}_n(X)$ (i.e., C_n is the union of finitely many locally closed sets). Then we define the measure

$$\mu(C) = [\pi_n(C)]\mathbb{L}^{-d(n+1)} \in \mathcal{M}_{\|}.$$

Let $f\colon X \to \mathbb{A}^1_k$ be a map defining a hypersurface in X. The **motivic zeta function** is

$$Z_{\mathrm{mot}}(f,s) = \int_{\mathcal{L}_\infty(X)} \mathbb{L}^{-\mathrm{ord}_t f \cdot s} = \sum_{i \geq 0} \mu(\mathrm{ord}_t f^{-1}(i)) \cdot \mathbb{L}^{-is} \in \mu_k[[\mathbb{L}^{-s}]].$$

By taking $t = \mathbb{L}^{-s}$, we recover the ordinary zeta function. Here ord_t denotes the **t-order**.
Upshot: the Grothendieck ring of varieties is the world in which motivic integration takes place, and there are relations between motivic integrals and zeta functions. The construction of motivic integrals was motivated by Kontsevich's proof that birationally equivalent Calabi–Yau varieties have the same Hodge numbers.

7.3 The ring structure of $\mathsf{K}_0(\mathsf{Var}_k)$

Remark 7.6. $\mathsf{K}_0(\mathsf{Var}_k)$ is a badly behaved ring:

- $\mathsf{K}_0(\mathsf{Var}_k)$ is infinite,
- $\mathsf{K}_0(\mathsf{Var}_k)$ is not Noetherian (Liu–Sebag '10),
- $\mathsf{K}_0(\mathsf{Var}_k)$ is not an integral domain (Poonen '02),
- $\mathsf{K}_0(\mathsf{Var}_k)/\mathbb{L} \cong \mathbb{Z}[\mathrm{SB}]$, where SB denotes stably birational equivalence classes of k-varieties (birational after multiplication by a large projective space),
- $\mathbb{L}$ is a zero divisor over $\mathbb{C}$ (Borisov '18). Borisov constructs varieties X, Y over $\mathbb{C}$ such that

  $$[X](\mathbb{L}^2 - 1)(\mathbb{L} - 1)\mathbb{L}^7 = [Y](\mathbb{L}^2 - 1)(\mathbb{L} - 1)\mathbb{L}^7$$

  but $[X] \neq [Y]$.

8 Talk 8: Annihilator of the Lefschetz Motive (D. Zack Garza)

Reference: [Zak17]

8.1 Background and Motivating questions

Remark 8.1. Let's begin by getting a sense of where we are and where we are headed:

- Yesterday we discussed classical scissors congruence.
- The main theme of today is going from scissors congruence to $\mathsf{K}-$theory; that is, how can we encode and detect scissors congruence in the language of $\mathsf{K}-$theory? One approach we've seen uses assemblers to enrich the classical Grothendieck group to a spectrum, and we've seen how classical motivic measures can be formulated in this setting.
- Tomorrow and for the next few days, we'll be studying how to go from $\mathsf{K}-$theory back to scissors congruence; that is, what kind of cut-and-paste information is encoded in K_0 and higher K_i? We will discus enriching motivic measures, generalizing assemblers to other cut-and-paste problems, and working towards topological approaches to a generalized variant of Hilbert's 3rd problem.

Although we are now likely familiar with most of the objects that will appear here, there are some subtle differences in conventions that are worth highlighting:

Definition 8.2 (Varieties)**.** Let k be a field and Var_k be the category of **varieties** over k, which we will take to mean reduced separated schemes of finite-type over the point $\operatorname{Spec} k$. We will say two varieties X, Y are **isomorphic** if and only if they are isomorphic in Sch_k, and will denote this by $X \cong Y$.

Warnings 8.3. There is a subtlety in the definition of the category of schemes: a morphism (and hence an isomorphism) of schemes $(X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is *not* simply a morphism of arbitrary ringed spaces, which would be a pair (F, ϕ) where $F : X \to Y$ is a morphism of spaces and $\phi : \mathcal{O}_Y \to F_* \mathcal{O}_X$ is a morphism of sheaves, where F_* denotes the direct image. Instead, they are defined as maps $f_i : U_i \to V_i$ defined on open affine covers $\{U_i = \operatorname{Spec} R_i\}, \{V_i = \operatorname{Spec} S_i\}$ of X and Y respectively where each f_i is induced by a morphism of rings $S_i \to R_i$. Equivalently, morphisms of schemes can be characterized as morphisms of *locally* ringed spaces.

Definition 8.4 (Stratified spaces)**.** Let X be a topological space, and for $U, V \subseteq X$, write $Y = U \biguplus V$ for the *internal disjoint union*, which indicates that U and V may not necessarily be disjoint but that their intersection $U \cap V$ is measure zero (which for example occurs if the intersection is lower-dimensional). A

stratification of X is the data of a (internally disjoint) partition of X into locally closed subspaces $X = \biguplus_{i \in I} X_i$ indexed by a poset $(I, \leq)$. The subspaces X_i are referred to as **strata**, and we additionally require that for each $j \in I$,

$$\overline{X_j} \subset \biguplus_{i \leq j} X_i,$$

i.e. the closure of X_j in X is contained in the union of lower-index strata.

Definition 8.5 (The Grothendieck ring of varieties)**.** Let Sp be a category of spectra – concretely, one can take the category of symmetric spectra of simplicial sets along with its stable model structure with levelwise cofibrations. Let $\mathcal{V}_k$ to be the assembler whose objects are the objects of Var_k and whose morphisms are closed inclusions of varieties, or equivalently locally closed embeddings of schemes. Since the field k will be fixed in the statements of most theorems, we will suppress the base field and write $\mathcal{V}$.

Let $\mathsf{K}(\mathcal{V})$ be its associated K-theory spectrum. The group $\mathsf{K}_0(\mathcal{V}) := \pi_0 \mathsf{K}(\mathcal{V})$ has a ring structure and can be shown to coincide with the **Grothendieck ring of varieties** as in Michael's talk. We will write elements in this ring using square brackets, so if X is a variety, $[X]$ denotes its equivalence class in $\mathsf{K}_0(\mathcal{V})$.

Definition 8.6 (The Lefschetz motive and its annihilator)**.** The class of the affine line $\mathbb{A}^1 := \mathbb{A}^1_k$ in $\mathsf{K}_0(\mathcal{V})$ is referred to as **the Lefschetz motive** and denoted

$$\mathbb{L} := [\mathbb{A}^1_k] \in \mathsf{K}_0(\mathcal{V}),$$

where we suppress the dependence on the base field k. Since this is simply an element of a ring, we can define its annihilator in the usual way as

$$\mathrm{Ann}(\mathbb{L}) := \ker(\mathsf{K}_0(\mathcal{V}) \xrightarrow{\cdot \mathbb{L}} \mathsf{K}_0(\mathcal{V})),$$

where $\cdot\mathbb{L}$ is the map induced by the morphism of assemblers

$$\begin{aligned} F : \mathcal{V} &\to \mathcal{V} \\ X &\mapsto X \underset{k}{\times} \mathbb{A}^1_k \end{aligned}$$

Fact 8.7. It is an exercise in commutative algebra that $\mathbb{L}$ is a ring-theoretic zero divisor in $\mathsf{K}_0(\mathcal{V})$ if and only if $\mathrm{Ann}(\mathbb{L}) = 0$. A first step toward understanding equations in a ring might be understanding its zero divisors, and several motivating problems and conjectures concern whether or not $\mathbb{L}$ in particular is a zero divisor. As a convention, we will frame questions about zero divisors in terms of triviality of annihilators. Of particular interest will be when $\mathrm{Ann}(\mathbb{L})$ is trivial as one varies the ground field k.

Example 8.8 (Working with $\mathbb{L}$)**.** We saw in talk 7 how to work with certain elements in $\mathsf{K}_0(\mathcal{V})$ and some formulas involving $\mathbb{L}$. One can show the following identities:

- $[\mathbb{G}_m] := [\mathbb{A}^n \setminus \{0\}] = \mathbb{L} - [\mathrm{pt}]$,

- $[\mathbb{P}^1] = \mathbb{L} + [\mathrm{pt}]$,
- For $\mathcal{E} \to X$ a rank n vector bundle[10], $[\mathcal{E}] = [X] \cdot [\mathbb{A}^n] = [X] \cdot \mathbb{L}^n$.

The last example shows that $\mathsf{K}_0(\mathcal{V})$ does not distinguish between trivial and nontrivial bundles. [Bor15] profitably uses this fact and similar computations to prove that a cut-and-paste conjecture of Larsen-Lunts fails, which conjecturally has applications to rationality of motivic zeta functions.

Definition 8.9 (Birational varieties)**.** Two varieties X, Y are **birational** if and only there is an isomorphism of $\varphi : U \xrightarrow{\sim} V$ of nonempty dense[11] open subschemes. Note that φ need not extend to a well-defined function on all of X and Y, and does not generally imply $X \cong Y$.

Remark 8.10. It is a standard convention to denote such a birational morphism defined on $U \subseteq X$ and $V \subseteq Y$ as $X \dashrightarrow Y$; here I will use the suggestive notation $X \overset{\sim}{\dashrightarrow} Y$ as a reminder that birational varieties are meant to be "almost" isomorphic. Why is this? In equations, a birational morphism φ is given not by polynomial equations but rather by rational functions, which allows denominators and introduces poles or a branch locus – generally in the complements $X \setminus U$ and $Y \setminus V$ respectively. These exceptional singular loci are meant to be "small" in some sense.

This weakened notion of isomorphism turns out to be the right way to study the **minimal model program**, an active area of current research which aims for a full classification of varieties up to some notion of equivalence, along with an understanding of particularly nice[12] "minimal" representatives in each class. This is of course an extremely difficult problem, but moving into the world of birational morphisms yields a much more tractable problem since the exceptional loci can often be stratified and cut into smaller pieces to study.

Definition 8.11 (Stable birationality)**.** Two varieties X, Y are **stably birational** if and only if there is a birational isomorphism

$$X \times \mathbb{P}^N \overset{\sim}{\dashrightarrow} Y \times \mathbb{P}^M$$

for some N, M large enough.

Remark 8.12. Many interesting invariants of birational geometry are in fact *stable* birational invariants. Some examples include:

- The Hodge number

$$h^{0,1}(X) = \dim_{\mathbb{C}} H^{0,1}(X^{\mathrm{an}})$$

where X^{an} as the analytic space associated to X and $H^{p,q}(X^{\mathrm{an}}) \coloneqq H^0(X^{\mathrm{an}}; \Omega^1_{X^{\mathrm{an}}})$,

[10] Here, a vector bundle over a variety X means a Zariski-locally trivial fibration over X with fibers isomorphic to $\mathbb{A}^n$.

[11] In fact, any nonempty open subset $U \subseteq X$ is automatically dense in X in the Zariski topology.

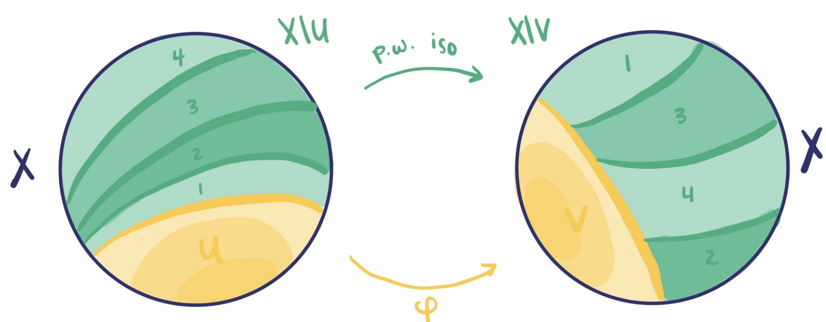
[12] Smooth, or singular with very well-understood singularities.

- the (analytic) fundamental group $\pi_1(X^{\mathrm{an}})$, and
- the zeroth Chow group $\mathrm{CH}_0(X)$.

A recent exposition of other applications of stable birationality is given in [Voi16].

Definition 8.13 (Piecewise isomorphisms)**.** Two varieties X, Y are **piecewise isomorphic** if and only if there exist stratifications $X = \biguplus_{i\in I} X_i$ and $Y = \biguplus_{i\in I} Y_i$ with each $X_i \cong Y_i$. Since we will be working with several notions of isomorphism, we will denote piecewise isomorphisms by $X \underset{\mathrm{pw}}{\cong} Y$.

Remark 8.14. This definition of a piecewise isomorphism is meant to capture the notion of cut-and-paste equivalence of varieties. To see how this relates to K-theory, note that if X and Y are piecewise isomorphism, then their classes are equal in $\mathsf{K}_0(\mathcal{V})$. On the other hand, if X and Y are birational, it is not generally the case that their classes are equal in $\mathsf{K}_0(\mathcal{V})$. However, if there is a birational morphism $X \dashrightarrow Y$ defined on $U \subseteq X$ and $V \subseteq Y$ and one *additionally* requires that $X \setminus U \cong Y \setminus V$, then X and Y are in fact piecewise isomorphic and thus have equal classes in $\mathsf{K}_0(\mathcal{V})$.



There are two broad motivational questions we would like to consider:

Question 8.15 (Motivating question 1)**.** When is the canonical ring localization morphism $\mathsf{K}_0(\mathcal{V}) \to \mathsf{K}_0(\mathcal{V})[1/\mathbb{L}]$ injective? In particular, when can equations in the localization be pulled back to valid equations in the original ring?

More philosophically, what does equality in $\mathsf{K}_0(\mathcal{V})$ actually *mean* geometrically? What geometric information is the Grothendieck ring capturing, and what conclusions can be drawn from equations in this ring?

Question 8.16 (Motivating question 2)**.** When is $\mathrm{Ann}(\mathbb{L})$ nonzero?

Remark 8.17. [Zak17] poses and answers several structural questions as a way to shed light on these:

Fact 8.18. There is a filtration on $\mathsf{K}_0(\mathcal{V}_k)$ such that the associated graded is

$$\mathsf{gr}_n \mathsf{K}_0(\mathcal{V}) = \mathrm{im}\left(\frac{\mathbb{Z}\Big[X \;\Big|\; \dim X \leq n\Big]}{([X] = [Y] + [X \setminus Y])} \xrightarrow{\psi_n} \mathsf{K}_0(\mathcal{V}) \right).$$

Question 8.19 (Structural question 1, Gromov)**.** If $U, V \hookrightarrow X$ with $X \setminus U \cong X \setminus V$, how far are U and V from being birational? If $X = Y$, can every birational automorphism $\phi : X \xrightarrow{\sim} X$ be extended to a piecewise isomorphism $\tilde{\phi} : X \underset{\text{pw}}{\cong} X$? This can equivalently be restated as a question about injectivity of the maps ψ_n above, where failure of injectivity at a particular n indicates extra relations in $\mathsf{K}_0(\mathcal{V})$ coming from classes of higher-dimensional varieties.

Conjecture 8.20 (A cut-and-paste conjecture of Larsen-Lunts)**.** If $[X] = [Y]$ is an equality the Grothendieck ring K_0, then there is a piecewise isomorphism $X \underset{\text{pw}}{\cong} Y$.

Remark 8.21. This conjecture is now known to be false – Borisov and Karzhemanov construct counterexamples for fields k that embed in $\mathbb{C}$, and [Zak17] shows that this additionally fails for a wider class of *convenient*[13] fields.

Conjecture 8.22. This is almost true, and the only obstructions come from $\mathrm{Ann}(\mathbb{L})$.

Conjecture 8.23. For certain varieties, equality $[X] = [Y]$ in the Grothendieck ring implies that X, Y are **stably birational**.

Remark 8.24. For the second motivating question, why might one care about this *particular* ring-theoretic property? Recall that this condition is equivalent to the injectivity of the map $\cdot\mathbb{L}$, so one answer is that having a nonzero annihilator allows cancellation of $\mathbb{L}$ in equations. Thus computations like the following can be carried out:

$$[X] \cdot \mathbb{L} = [Y] \cdot \mathbb{L} \implies ([X] - [Y]) \cdot \mathbb{L} = 0 \overset{\mathrm{Ann}(\mathbb{L})=0}{\implies} [X] - [Y] = 0 \implies [X] = [Y],$$

and so equality "up to a power of $\mathbb{L}$" implies honest equality. A separate motivation comes from the purely algebraic fact that the localization morphism $R \to S^{-1}R$ for a multiplicative set S is injective precisely when S does not contain zero divisors, and so if $\mathrm{Ann}(\mathbb{L}) = 0$ then $\mathsf{K}_0(\mathcal{V}) \hookrightarrow \mathsf{K}_0(\mathcal{V})[1/\mathbb{L}]$ is injective.

The latter ring appears in conjectures concerning rationality of motivic zeta functions $\zeta_X(t)$. The recent paper [LL20] exhibits a K3 surface X in such that $\zeta_X(t)$ is *not* rational over $\mathsf{K}_0(\mathcal{V})$, and discuss the possibility of its rationality as a formal power series in $\mathsf{K}_0(\mathcal{V})[1/\mathbb{L}]$ instead.

Answer 8.25. [Bor15] and [Kar14] partially answer this question by showing that $\mathbb{L}$ generally **is** a zero divisor, witnessed by explicit constructions of elements that are equal in $\mathsf{K}_0(\mathcal{V})$ but not piecewise isomorphic, thus yielding nontrivial elements in $\mathrm{Ann}(\mathbb{L})$. Seemingly coincidentally, their construction also produces elements in $\ker \psi_n$, and so a natural question is whether or not this is actually a coincidence at all.

Proposition 8.1 (Borisov)**.** *The cut-and-paste conjecture of Larsen and Lunts is false.*

[13] This is a technical condition to be described later.

Proof. This is proved in [Bor15, Theorem 2.13]. There is a certain pair of "mirror" varieties X_W and Y_W[14] which are provably **not** birational and for which stable birationality would imply birationality. One starts with an equality in $\mathsf{K}_0(\mathcal{V})$, and toward a contradiction supposes that equality in the Grothendieck ring implies piecewise-isomorphism. Several properties of bundles over these varieties are used to make the following series of computations:

$$\begin{aligned}
&[X_W]\left(\mathbb{L}^2-1\right)(\mathbb{L}-1)\mathbb{L}^7 = [Y_W]\left(\mathbb{L}^2-1\right)(\mathbb{L}-1)\mathbb{L}^7 \\
\Longrightarrow\ &[\mathrm{GL}_2(\mathbb{C})\times\mathbb{C}^6\times X_W] = [\mathrm{GL}_2(\mathbb{C})\times\mathbb{C}^6\times Y_W] \\
\Longrightarrow\ &\mathrm{GL}_2(\mathbb{C})\times\mathbb{C}^6\times X_W \underset{\mathrm{pw}}{\cong} \mathrm{GL}_2(\mathbb{C})\times\mathbb{C}^6\times Y_W \qquad \text{if Larsen-Lunts is true} \\
\Longrightarrow\ &X_W\times\mathrm{GL}_2(\mathbb{C})\times\mathbb{C}^6 \overset{\sim}{\dashrightarrow} Y_W\times\mathrm{GL}_2(\mathbb{C})\times\mathbb{C}^6 \\
&\Longrightarrow\ X_W \overset{\sim_{\mathrm{Stab}}}{\dashrightarrow} Y_W \qquad \text{i.e. } X_W, Y_W \text{ are stably birational} \\
&\Longrightarrow\ X_W \overset{\sim}{\dashrightarrow} Y_W,
\end{aligned}$$

concluding that X_W and Y_W are birational, a contradiction. □

Question 8.26. How and why are $\mathrm{Ann}(\mathbb{L})$ and $\ker\psi_n$ related? [Zak17] gives a precise answer.

8.2 Outline of Results

Slogan 8.27. The following are some slogans for what's shown in [Zak17], to give you some feeling for what might be true:

- **Theorem A:** There is a stable (filtered) homotopy type $\mathsf{K}(\mathcal{V})$ whose associated graded *spectrum* $\mathsf{gr}\,\mathsf{K}(\mathcal{V})$ is simpler than the the associated graded *ring* $\mathsf{gr}\,\mathsf{K}_0(\mathcal{V})$.
- **Theorem B**: The associated spectral sequence[15] is an obstruction theory for birational automorphisms extending to piecewise isomorphisms, and the spectral sequence detects $\ker\psi_n$ for various n.
- **Theorem C:** The motivating questions 1 and 2 are precisely linked: elements in $\mathrm{Ann}(\mathbb{L})$ yield elements in $\ker(\psi_n)$.
- **Theorem D**: There is a partial characterization of $\mathrm{Ann}(\mathbb{L})$ in terms of varieties satisfying certain equations in $\mathsf{K}_0(\mathcal{V})$ which are not piecewise isomorphic.

[14] Roughly speaking, these are smooth derived-equivalent Calabi-Yau threefolds, see the *Pfaffian-Grassmannian correspondence*.

[15] That is, the spectral sequence naturally associated to a filtered spectrum. How exactly this is constructed is spelled out in [Zak17, Section 2].

- **Theorem E**: $\mathsf{K}_0(\mathcal{V}) \mod \mathbb{L}$ completely captures stable birational geometry: there is an isomorphism of abelian groups[16]

$$\mathsf{K}_0(\mathcal{V}) / \langle \mathbb{L} \rangle \cong \mathbb{Z}[\mathrm{SB}],$$

  where SB is the set of stable birational equivalence classes of varieties.

Moreover, a main conclusion is that elements in $\mathrm{Ann}(\mathbb{L})$ *always* produce elements in $\ker \psi_n$. We'll now look at these theorems in more detail.

8.3 Theorems and proof sketches

Theorem 8.28 ([Zak17] Theorem A)**.** *There is a homotopical enrichment of* $\mathsf{K}_0(\mathcal{V})$ *with a simple associated graded. Let*

- $\mathcal{V}^{(n)}$ *be the nth filtered assembler of $\mathcal{V}$ generated by varieties of dimension* $d \leq n$*,*
- $\mathrm{Aut}_k\, k(X)$ *be the group of birational automorphisms of the variety X,*
- B_n *be the set of birational isomorphism classes of varieties of dimension* $d = n$*.*

There is a spectrum $\mathsf{K}(\mathcal{V})$ such that $\mathsf{K}_0(\mathcal{V}) \coloneqq \pi_0 \mathsf{K}(\mathcal{V})$ coincides with the previously defined Grothendieck group of varieties, and $\mathcal{V}^{(n)}$ induces a filtration on $\mathsf{K}(\mathcal{V})$ such that

$$\mathsf{gr}_n \mathsf{K}(\mathcal{V}) = \bigvee_{[X] \in B_n} \Sigma^\infty_+ \mathbf{B}\mathrm{Aut}_k\, k(X),$$

with an associated spectral sequence

$$E^1_{p,q} = \bigvee_{[X] \in B_n} (\pi_p \Sigma^\infty \mathbf{B}\mathrm{Aut}_k\, k(X) \oplus \pi_p \mathbb{S}) \Rightarrow \mathsf{K}_p(\mathcal{V})$$

Remark 8.29. Note that the $p = 0$ column converges to $\mathsf{K}_0(\mathcal{V})$.

Proof.
- Define $\mathcal{V}^{(n.n-1)} = \mathsf{Var}_k^{\dim = n} \cup \{\emptyset\}$, the varieties of dimension *exactly* n.
- Use [Zak17, Theorem 1.8] to extract cofibers in the filtration and identify the associated graded:

[16] This result was previously known, and the significance is that this can now be proved using homotopy-theoretic techniques.

$$
\begin{tikzcd}
\mathsf{K}(\mathcal{V}) \\
\vdots \arrow[u, hook] \\
\mathsf{K}(\mathcal{V}^{(n)}) \arrow[u, hook] \arrow[r, two heads] & \color{red}{\mathsf{K}(\mathcal{V}^{(n-1)})} \\
\mathsf{K}(\mathcal{V}^{(n-1)}) \arrow[u, hook] \arrow[r, two heads] & \color{red}{\mathsf{K}(\mathcal{V}^{(n-1,n-2)})} \\
\vdots \arrow[u, hook] \\
\mathsf{K}(\mathcal{V}^{(2)}) \arrow[u, hook] \arrow[r, two heads] & \color{red}{\mathsf{K}(\mathcal{V}^{(2,1)})} \\
\mathsf{K}(\mathcal{V}^{(1)}) \arrow[u, hook] \\
\text{Fil} & \text{gr}
\end{tikzcd}
$$

- Finish by a magic computation:

$$
\begin{aligned}
\mathsf{K}(\mathcal{V}^{(n,n-1)}) &\simeq \tilde{\mathsf{K}}(\mathcal{V}^{(n,n-1)}) \\
&\simeq \mathsf{K}(\mathsf{C}) \\
&\simeq \mathsf{K}\left(\bigvee_{\alpha \in B_n} \mathsf{C}_{X_\alpha}\right) \\
&\simeq \bigvee_{\alpha \in B_n} \mathsf{K}(\mathsf{C}_{X_\alpha}) \\
&\cong \bigvee_{\alpha \in B_n} \Sigma^\infty_+ \mathbf{B}\mathrm{Aut}_k k(X_\alpha) \qquad \text{Zak17a} \\
&\coloneqq \bigvee_{\alpha \in B_n} \Sigma^\infty_+ \mathbf{B}\,\mathrm{Aut}(\alpha),
\end{aligned}
$$

where

- $\tilde{\mathsf{K}}(\mathcal{V}^{(n,n-1)})$: the full subassembler of irreducible varieties.

  - **Why the reduction works:** general theorem [Zak17, Theorem 1.9] on subassemblers with enough disjoint open covers

- $\mathsf{C} \leq \mathcal{V}^{(n,n-1)}$: subvarieties of some X_α representing some α, as α ranges over B_n.

  - **Why the reduction works:** apply [Zak17, Theorem 1.9] again

- C_{X_α} is the subassembler of only those varieties admitting a (unique) morphism to X_α for a fixed α.

  - **Why the reduction works:** each nonempty variety admits a morphism to exactly one X_α representing some α – otherwise, if $X \mapsto X_\alpha, X_\beta$ then X_α and X_β are forced to be birational (the morphisms are inclusions of dense opens) implying $\alpha = \beta$

- $\mathrm{Aut}(\alpha) \coloneqq \mathrm{Aut}_k\, k(X)$ for any X representing $\alpha \in B_n$.

Note that much of this proof amounts to repeated application of dévissage. □

Theorem 8.30 ([Zak17] Theorem B)**.** *There exists nontrivial differentials d_r from column 1 to column 0 in some page of E^* $\iff$ $\cup_n \ker \psi_n \neq 0$ (ψ_n has a nonzero kernel for some n). More precisely, $\varphi \in \mathrm{Aut}_k k(X)$ extends to a piecewise automorphism if and only if $d_r[\varphi] = 0 \quad \forall r \geq 1$.*

Remark 8.31. Before proving this result, it is helpful to look at the actual spectral sequence. The following is the the E^1 page:

$$
\begin{tikzcd}
q & & & \\
& \vdots & \vdots & \\
\mathrm{Fil}_n & \mathsf{K}_0(\mathcal{V}^{(n,n-1)}) & \mathsf{K}_1(\mathcal{V}^{(n,n-1)}) \arrow[dl, bend right, "d_1"'] \arrow[ddl, dashed, bend right] \arrow[dddl, bend right, "d_n"] & \cdots \\
\mathrm{Fil}_{n-1} & \mathsf{K}_0(\mathcal{V}^{(n-1,n-2)}) & \mathsf{K}_1(\mathcal{V}^{(n-1,n-2)}) & \cdots \\
\vdots & \vdots & \vdots & \\
\mathrm{Fil}_0 & \mathsf{K}_0(\mathcal{V}^{(1,0)}) & \mathsf{K}_1(\mathcal{V}^{(1,0)}) & \cdots \\
& \pi_0 & \pi_1 & p
\end{tikzcd}
$$

To identify the terms, one carries out a short computation:

$$
\begin{aligned}
\mathsf{K}_p(\mathcal{V}^{(n,n-1)}) &\coloneqq \pi_p \mathsf{K}(\mathcal{V}^{(n,n-1)}) \\
&\simeq \pi_p \bigvee_{\alpha \in B_n} \Sigma^\infty_+ \mathbf{B}\,\mathrm{Aut}(\alpha) \\
&\cong \bigoplus_{\alpha \in B_n} \pi_p \Sigma^\infty_+ \mathbf{B}\,\mathrm{Aut}(\alpha).
\end{aligned}
$$

Now using that $\pi_p \Sigma^\infty_+ \mathbf{B} G$ is $\mathbb{Z}$ for $p = 0$ and $G^{\mathrm{ab}} \oplus C_2$ for $p = 2$, we have the following:

$$\begin{tikzcd}[column sep=small]
q & & & & \\
& & \vdots & \vdots & \\
\mathrm{Fil}_n & & \bigoplus_{\alpha\in B_n}\mathbb{Z} & \bigoplus_{\alpha\in B_n}\mathrm{Aut}(\alpha)^{\mathrm{ab}}\oplus C_2 \arrow[dl, bend right, "d_1"'] \arrow[ddl, dashed, bend right] \arrow[dddl, bend right, "d_n"] & \cdots \\
\mathrm{Fil}_{n-1} & & \bigoplus_{\alpha\in B_{n-1}}\mathbb{Z} & \bigoplus_{\alpha\in B_{n-1}}\mathrm{Aut}(\alpha)^{\mathrm{ab}}\oplus C_2 & \cdots \\
\vdots & & \vdots & \vdots & \\
\mathrm{Fil}_0 & & \bigoplus_{\alpha\in B_0}\mathbb{Z} & \bigoplus_{\alpha\in B_0}\mathrm{Aut}(\alpha)^{\mathrm{ab}}\oplus C_2 & \cdots \\
& & \pi_0 & \pi_1 & & p
\end{tikzcd}$$

Lemma 8.32 ([Zak17] Lemma 3.2)**.** *Note that there is a boundary map ∂ coming from the connecting map in the LES in homotopy of a pair for the filtration. If $\varphi \in \mathrm{Aut}(\alpha)$ for $\alpha \in B_q$ is represented by $\varphi : U \to V$ then*

$$\partial[\varphi] = [X \setminus V] - [X \setminus U] \quad \in \mathsf{K}_0(\mathcal{V}^{(q-1)})$$

Proof of lemma.

- In general, $x \in \mathsf{K}_1(\mathcal{V}^{(q,q-1)})$ corresponds to the following data: X a variety, a dense open subset with two embeddings F and G, the two possible complements, where $\{X_i\}$ is a covering family over X where $\bigcup\limits_i X_i$ is a dense open subset of X, and the complements are of dimension at most $q-1$:

$$\begin{tikzcd}
& & Y = X \setminus \mathrm{im}(F) \arrow[dl] \\
\bigcup_i X_i \arrow[r, shift left, "F"] \arrow[r, shift right, "G"'] & X & \\
& & Z = X \setminus \mathrm{im}(G) \arrow[ul]
\end{tikzcd}$$

- [Zak17, Prop. 3.13] shows that for this data,

$$\partial[x] = [Z] - [Y] \in \mathsf{K}_0(\mathcal{V}^{(q-1)})$$

- For φ, we can represent it with the data:

$$\begin{tikzcd}
 & & & Y = X \setminus \operatorname{im}(\iota_U) \arrow[dl] \\
U \arrow[rr, "\iota_U"] \arrow[dr, "\varphi"'] & & X & \\
 & V \arrow[ur, "\iota_V"'] & & Z = X \setminus \operatorname{im}(\iota_V \circ \varphi) \arrow[ul]
\end{tikzcd}$$

- One can then conclude

$$\partial[\varphi] = [Z] - [Y] = [X \setminus V] - [X \setminus U].$$

□

Proof of theorem B ($\Longrightarrow$). Suppose φ extends to a piecewise automorphism.

- Then $[X \setminus U] = [X \setminus V] \in \mathsf{K}_0(\mathcal{V}^{q-1})$ since $X \setminus U \xrightarrow{\sim} X \setminus V$ by assumption
- By lemma 3.2 above,

$$\partial[\varphi] = [X \setminus V] - [X \setminus U] = 0$$

- [Zak17, Lemma 2.1] shows that d_1 and higher d_r are built using ∂, so $\partial(x) = 0 \implies d_r(x) = 0$ for all $r \geq 1$, yielding a permanent boundary.

□

Proof of theorem B, ($\Longleftarrow$). Suppose $d_r[\varphi] = 0$ for all $r \geq 1$.

- Since in particular $d_1[\varphi] = 0$, we have

$$[X \setminus U] = [X \setminus V] \in \mathsf{K}_0(\mathcal{V}^{(q,q-1)}),$$

since $d_1 = \partial \circ p$ for some map p.

- An inductive argument allows one to write $X = U_r \uplus X'_r = V_r \uplus Y'_r$ where

$$U_r \underset{\text{pw}}{\cong} V_r, \quad \dim X'_r, \dim Y'_r < n - r, \quad \partial[\varphi] = [Y'_r] - [X'_r]$$

- Take $r = n$ to get

$$\dim X'_n, \dim Y'_n < 0 \implies X'_n = Y'_n = \emptyset \quad \text{and} \quad X = U_n = V_n$$

- Then

$$\partial[\varphi] = [\emptyset] - [\emptyset] = 0 \implies \varphi \text{ extends.}$$

$\square$

Remark 8.33. A general remark on why $\partial[\varphi] = 0$ implies it extends:

- $\partial[\varphi]$ measures the failure of φ to extend to a piecewise isomorphism:

$$\partial[\varphi] = 0 \implies [X \setminus V] = [X \setminus U] \implies \exists \psi : X \setminus V \underset{\text{pw}}{\cong} X \setminus U$$

- If additionally $U \cong V$ then $\varphi \uplus \psi$ assemble to a piecewise automorphism of X.

Theorem 8.34 ([Zak17] Theorem C)**.** *Let k be a* ***convenient field****, e.g.* $\operatorname{ch} k = 0$*. Then $\mathbb{L}$ is a zero divisor in $\mathsf{K}_0(\mathcal{V}) \implies \psi_n$ is not injective for some n. In other words, for k convenient,*

$$\operatorname{Ann}(\mathbb{L}) \neq 0 \implies \bigcup_n \ker \psi_n \neq \emptyset.$$

Proof of theorem C.

- Strategy: contrapositive. Suppose $\ker \psi_n = 0$ for all n. There is a cofiber sequence

$$\mathsf{K}(\mathcal{V}) \overset{\cdot\mathbb{L}}{\hookrightarrow} \mathsf{K}(\mathcal{V}) \overset{\ell}{\twoheadrightarrow} \mathsf{K}(\mathcal{V}/\mathbb{L})$$

  where $\mathcal{V}/\mathbb{L}$ is a "cofiber assembler" [Zak17, Def 1.11].

- Take the associated lon exact sequence to identify $\ker(\cdot\mathbb{L})$ with $\operatorname{coker}(\ell)$:

$$\begin{tikzcd}
 & & \vdots \arrow[dll, dashed] \\
\mathsf{K}_1(\mathcal{V}) \arrow[r, "\cdot\mathbb{L}"] & \mathsf{K}_1(\mathcal{V}) \arrow[r, "\ell"] & \mathsf{K}_1(\mathcal{V}/\mathbb{L}) \arrow[dll, "\partial"] \\
\mathsf{K}_0(\mathcal{V}) \arrow[r, "\cdot\mathbb{L}"] & \mathsf{K}_0(\mathcal{V}) \arrow[r, "\ell"] & \mathsf{K}_0(\mathcal{V}/\mathbb{L})
\end{tikzcd}$$

- Reduce to analyzing

$$\operatorname{coker}(E^{\infty}_{1,q} \to \tilde{E}^{\infty}_{1,q})$$

  where $\tilde{E}$ is an auxiliary spectral sequence.

- Suppose all α extend, then all differentials from column 1 to column 0 are zero.

- The map $E^r \to \tilde{E}^r$ is surjective for all r on all components that survive to E^∞.
- All differentials out of these components are zero, so $E^\infty \twoheadrightarrow \tilde{E}^\infty$.
- Then $\mathsf{K}_1(\mathcal{V}) \overset{\ell}{\twoheadrightarrow} \mathsf{K}_1(\mathcal{V}/\mathbb{L})$, making $0 = \operatorname{coker}(\ell) = \ker(\cdot\mathbb{L})$ so $\mathbb{L}$ is not a zero divisor.

$\square$

Theorem 8.35 ([Zak17] Theorem D)**.** *Suppose that k is a* ***convenient*** *field. If $\chi \in \operatorname{Ann}(\mathbb{L})$ then $\chi = [X] - [Y]$ where*

$$\left[X \times \mathbb{A}^1\right] = \left[Y \times \mathbb{A}^1\right] \quad \textit{but } X \times \mathbb{A}^1 \underset{\text{pw}}{\not\cong} Y \times \mathbb{A}^1.$$

Thus elements in $\operatorname{Ann}(\mathbb{L})$ give rise to elements in $\bigcup_{n \geq 0} \ker \psi_n$.

Proof of theorem D.

- Let $\chi \in \ker(\cdot\mathbb{L})$ and pullback in the LES to $x \in \mathsf{K}(\mathcal{V}^{(n)}/\mathbb{L})$ where n is minimal among filtration degrees:

$$\begin{tikzcd}
 & & \vdots \arrow[dll, dashed] \\
\mathsf{K}_1(\mathcal{V}^{(n-1)}) \arrow[r, "\cdot\mathbb{L}"] & \mathsf{K}_1(\mathcal{V}^{(n)}) \arrow[r, "\ell"] & \mathsf{K}_1(\mathcal{V}^{(n)}/\mathbb{L}) \ni x \arrow[dll, "\partial"] \\
\mathsf{K}_0(\mathcal{V}^{(n-1)}) \arrow[r, "\cdot\mathbb{L}"] & \mathsf{K}_0(\mathcal{V}^{(n)}) \arrow[r, "\ell"] & \mathsf{K}_0(\mathcal{V}^{(n)}/\mathbb{L}) \\
\chi \arrow[r, mapsto] & 0 &
\end{tikzcd}$$

- Write $\partial[x] = [X] - [Y]$ with X, Y of minimal dimension.
- By [**LS10**],

$$\begin{aligned}
[X \times \mathbb{A}^1] = [Y \times \mathbb{A}^1] &\implies \dim X + 1 = \dim Y + 1 \\
&\implies \dim X = \dim Y = d
\end{aligned}$$

Claim 8.36. d is small: $d < n - 1$.

Note that we're done if this claim is true: proceed by showing X and Y are not piecewise isomorphic by showing $\ker \psi_n$ is nontrivial by a diagram chase. $\square$

Proof of claim. The proof boils down to a diagram chase, which roughly goes as follows:

$$
\begin{tikzcd}
 & 1 & 2 & \\
 & {[X]-[Y] \notin \operatorname{im}\partial^{(n-1)}} \arrow[r, mapsto, dashed] \arrow[ddd, mapsto, dotted, bend left] & {\mathbb{L}([X]-[Y]) \neq 0} \arrow[ddd, mapsto, dotted, bend left] & \\
\mathsf{K}_1(\mathcal{V}^{(n-1)}/\mathbb{L}) \arrow[r, "\partial^{(n-1)}"] \arrow[d] & \mathsf{K}_0(\mathcal{V}^{(n-2)}/\mathbb{L}) \arrow[r, "\cdot\mathbb{L}_{n-2}"] \arrow[d, "i_*^{n-1}"] & \mathsf{K}_0(\mathcal{V}^{(n-1)}) \arrow[d, "i_*^{n-1}"] & 4 \\
\mathsf{K}_1(\mathcal{V}^{(n)}/\mathbb{L}) \arrow[r, "\partial^{(n)}"] & \mathsf{K}_0(\mathcal{V}^{(n-1)}/\mathbb{L}) \arrow[r, "\cdot\mathbb{L}_{n-1}"] & \mathsf{K}_0(\mathcal{V}^{(n)}) & \\
 & {i_*^{n-1}([X]-[Y]) \in \operatorname{im}\partial^{(n)}} \arrow[r, mapsto, dotted] & 0 & \\
 & 3 & &
\end{tikzcd}
$$

1. $[X]-[Y] \notin \operatorname{im}(\partial)$ by the minimality of n for x, noting $\partial[x] = [X]-[Y]$.
2. By exactness $\operatorname{im}\partial = \ker(\cdot\mathbb{L})$, so $\mathbb{L}([X]-[Y]) \neq 0$.
3. By choice of n, $i_*(\mathbb{L}([X]-[Y])) \in \operatorname{im}\partial = \ker(\cdot\mathbb{L})$ in bottom row, so $\mathbb{L}([X]-[Y]) = 0$ in bottom-right.
4. Commutativity forces $\mathbb{L}([X]-[Y]) \in \ker i_*^{n-1}$.

Thus $\mathbb{L}([X]-[Y])$ corresponds to an element in $\ker\psi_n$.

□

Theorem 8.37 ([Zak17] Theorem E)**.** *There is an isomorphism*

$$\mathsf{K}_0(\mathcal{V}_{\mathbb{C}})/\langle\mathbb{L}\rangle \xrightarrow{\sim} \mathbb{Z}[\mathsf{SB}_{\mathbb{C}}] \qquad \in \mathbb{Z}\text{-}\mathsf{Mod}.$$

Remark 8.38. Proof: omitted.

8.4 Closing Remarks

Remark 8.39. What we've accomplished: establishing a precise relationship between questions 1 and 2.

Question 8.40. Some currently open questions:

- What fields are convenient?

- What is the associated graded for the filtration induced by ψ_n?
- Is there a characterization of $\mathrm{Ann}(\mathbb{L})$?
- (Interesting) What is the kernel of the localization $\mathsf{K}_0(\mathcal{V}) \to \mathsf{K}_0(\mathcal{V})[\frac{1}{\mathbb{L}}]$?
- Does ψ_n fail to be injective over *every* field k?

Conjecture 8.41. There is a correction to Question 1 cpncerning $\ker \psi_n$ which may be true: let X, Y be varieties over a convenient field with $[X] = [Y$. Then there exist varieties X', Y' such that

- $[X'] \neq [Y']$
- $[X' \times \mathbb{A}^1] = [X'] \cdot \mathbb{L} = [Y'] \cdot \mathbb{L} = [Y' \times \mathbb{A}^1]$
- $X \coprod X' \times \mathbb{A}^1 \underset{\text{pw}}{\cong} Y \coprod Y' \times \mathbb{A}^1$

Remark 8.42. If the conjecture holds, if X, Y are not birational but are *stably* birational, then the error of birationality is measured by a power of $\mathbb{L}$.

Contingent upon this conjecture, one might hope to show

$$[X] \equiv [Y] \bmod \mathbb{L} \implies X \overset{\sim_{\text{Stab}}}{\dashrightarrow} Y,$$

so that the equality in the quotient ring completely captures stable birational geometry.

9 Talk 9: SW-categories (Eunice Sukarto)

Reference: [Cam19].

9.1 Recollections

Remark 9.1. Let k be a field and $\mathsf{Var}_{/k}$ be the category of finite type separated schemes over $\operatorname{Spec} k$, and recall

$$\mathsf{K}_0(\mathsf{Var}_{/k}) = \mathbb{Z}\left[\left\{[X] \;\middle|\; X \in \mathsf{Var}_{/k}\right\}\right] / \sim$$

where the relations are given by $[Y] = [X] + [Y \setminus X]$ for closed immersions $X \hookrightarrow Y$. Inna constructs a spectrum for this using assemblers, but an issue is that it is hard to map out of this spectrum.

Our goal is to construct $\mathsf{K}(\mathsf{Var}_k)$, a spectrum with the property that $\pi_0(\mathsf{K}(\mathsf{Var}) = \mathsf{K}_0(\mathsf{Var}_k)$. That way we can define $K_n(\mathsf{Var}_k) = \pi_n \mathsf{K}(\mathsf{Var}_k)$. Today we'll consider Jonathan's construction: a different construction of the spectrum, and an additivity theorem which provides a delooping and plays a role in exhibiting an $\mathbb{E}_\infty$ structure.

Slogan 9.2. K-theory is the universal machine that splits cofiber sequences.

Remark 9.3. For $\mathsf{Var}_{/k}$, what are the cofiber sequences? Cofibrations will be closed immersions, and cofiber sequences will be pushouts along the terminal object. We want $X \hookrightarrow Y \leftarrow Y \setminus X$, but the Waldhausen construction doesn't necessarily go through. The idea is to use *subtraction sequences* $\hookrightarrow \bullet \overset{\circ}{\leftarrow}$ instead of cofibration sequences used in the Waldhausen construction.

First let us review the classical Waldhausen $S_\bullet$-construction. In a later talk we'll see that this appropriately generalizes both algebraic K-theory and Quillen's Q-construction.

9.2 Waldhausen's $S_\bullet$-construction

Remark 9.4. Input: a Waldhausen category C with a zero object, cofibrations $\hookrightarrow$, and weak equivalences $\xrightarrow{\sim}$ such that $\mathrm{pt} \hookrightarrow X$ is a cofibration, isomorphisms of C are both cofibrations and weak equivalences, and the induced map of pushouts of the following two rows are weak equivalences:

$$\begin{tikzcd} D \arrow[d, dashed, "\sim"] & C \arrow[l, hook'] \arrow[r] \arrow[d, dashed, "\sim"] & E \arrow[d, dashed, "\sim"] \\ D' & C' \arrow[l, hook'] \arrow[r] & E' \end{tikzcd}$$

The output is $\mathsf{K}(\mathsf{C})$, the algebraic K-theory spectrum, where $\pi_0(\mathsf{K}(\mathsf{C})) = \mathsf{K}_0(\mathsf{C})$. The general blueprint is to first construct a space where π_1 is K_0, then take loops to shift. The complex is as described as follows:

- 0-simplices: pt
- 1-simplices: for all X, pt $\xrightarrow{X}$ pt
- 2-simplices: for each $X \hookrightarrow Y \twoheadrightarrow Y/X$, a triangle

$$\begin{tikzcd}
& \mathrm{pt} & \\
\mathrm{pt} \arrow[ur, "Y\setminus X"] \arrow[rr, "Y"'] & & \mathrm{pt} \arrow[ul, "X"']
\end{tikzcd}$$

- If we have

$$\begin{tikzcd}
X \arrow[r, hook] & Y \arrow[r, hook] \arrow[d, two heads] & Z \arrow[d, two heads] \\
& Y/X \arrow[r, hook] \arrow[ur, phantom, "\ulcorner"] & Z/X \arrow[d, two heads] \\
& & Z/Y
\end{tikzcd}$$

- Since

$$\begin{aligned} Z = X + Z/X &= X + (Y/X + Z/Y) \\ Y + Z/Y &= (X + Y/X) + Z \setminus Y, \end{aligned}$$

  we'll force these to be homotopic with a 3-simplex:

$$\begin{tikzcd}
& \bullet & \\
\bullet \arrow[ur, "Z/XY"] \arrow[rr, "Z/X"] \arrow[dr, "Z"'] & & \bullet \arrow[ul, "Y/X"'] \\
& \bullet \arrow[uu, "Y"'] \arrow[ur, "X"'] &
\end{tikzcd}$$

and similarly, for any sequence $X_1 \hookrightarrow X_2 \hookrightarrow \cdots \hookrightarrow X_n$ we get an n-simplex capturing such a relationship.

To formalize this, define an arrow category $\mathsf{Ar}[n]$ whose objects are pairs (i, j) with $0 \le i \le j \le n$ with morphisms $(i, j) \to (i', j')$ iff $i \le i', j \le i'$. Let $S_n\mathsf{C} \le \mathsf{Fun}(\mathsf{Ar}[n], \mathsf{C})$ be the full subcategory where consecutive horizontal maps are cofibration sequences and the squares form pushouts. Below is a diagram for $S_3\mathsf{C}$:

$$\begin{tikzcd}
\mathrm{pt} \arrow[r, hook] & \bullet \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near end] & \bullet \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near end] & \bullet \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near end] & \bullet \arrow[d] \\
& \mathrm{pt} \arrow[r, hook] & \bullet \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near end] & \bullet \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near end] & \bullet \arrow[d] \\
& & \mathrm{pt} \arrow[r, hook] & \bullet \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near end] & \bullet \arrow[d] \\
& & & \mathrm{pt} \arrow[r, hook] & \bullet \arrow[d] \\
& & & & \mathrm{pt}
\end{tikzcd}$$

Define $S_\bullet\check{C}$ to be the simplicial category $[n] \to S_n\mathsf{C}$ whose faces and degeneracies are obtained by inserting/deleting identities. Define $wS_n\mathsf{C}$ to be the simplicial subcategory generated by weak equivalences, and define the bisimplicial set $w.S_\bullet\mathsf{C} = \{NwS_\bullet\mathsf{C}\}$. The algebraic K-theory space is defined as

$$\mathsf{K}(\mathsf{C}) := \Omega|w.S_\bullet\mathsf{C}|.$$

The additivity theorem implies $\mathsf{K}(\mathsf{C})$ is an infinite loop space and thus a spectrum.

Remark 9.5. For the $\tilde{S}_\bullet$ construction the input will be a category with *subtraction* (i.e. subtraction sequences and pullbacks) and outputs a spectrum $\mathsf{K}(\mathsf{C})$. Some categories that this applies to: subtraction categories and SW categories, where a product structure on the category yields an $\mathbb{E}_\infty$ structure on the spectrum.

Remark 9.6. Categories with subtractions have the following data:

- Initial objects $\emptyset$,
- Cofibrations $\hookrightarrow$,
- Fibrations $\overset{\circ}{\to}$,
- Pullbacks that satisfy base change,
- Isomorphisms which are both cofibrations and fibrations
- Subtraction sequences satisfying some nice properties, e.g. $A \to A\coprod B \leftarrow B$. Denote this $\hookrightarrow\overset{\circ}{\to}$.

Example 9.7. $\mathsf{Sch}_{/X}, \mathsf{Var}_{/X}$ are subtraction categories, where cofibrations are closed immersions and fibrations are closed immersions.

Definition 9.8 (Subtractive categories)**.** A **subtractive category** is a category with subtractions such that the following squares exists and are Cartesian:

$$\begin{tikzcd}
\bullet \arrow[r, hook] \arrow[d, hook] & \bullet \arrow[d, hook] \\
\bullet \arrow[r, hook] \arrow[ur, phantom, "\ulcorner" very near end] & \bullet
\end{tikzcd}$$

Remark 9.9. Examples include $\mathsf{Sch}_{/X}, \mathsf{Var}_{/X}$, but not *smooth* schemes since pushouts can introduce singularities.

Remark 9.10. To a subtractive category C we can define a category $F_1^+\mathsf{C}$ whose objects are subtraction sequences and morphisms are cartesian squares. Note that F_1^+ is itself subtractive, and functors $(X \hookrightarrow Y \overset{\circ}{\leftarrow} Z) \to X, Y, Z$ respectively which takes in a from $F_1^+\mathsf{C} \to \mathsf{C}$.

Definition 9.11. SW-categories: Subtractive Waldhausen categories plus conditions, e.g. isomorphism are weak equivalences and compatibility with subtraction sequences. Then we can define the $\tilde{S}_\bullet$ construction – let $\tilde{\mathsf{Ar}}[n]$ be the arrow category where every rectangle with $\emptyset$ in the bottom-left is a Cartesian square:

$$\begin{tikzcd}
\emptyset \arrow[r, hook] & \bullet \arrow[r, hook] & \bullet \arrow[r, hook] & \bullet \arrow[r, hook] & \bullet \\
& \emptyset \arrow[r, hook] \arrow[u, "o"] & \bullet \arrow[r, hook] \arrow[u, "o"] & \bullet \arrow[r, hook] \arrow[u, "o"] & \bullet \arrow[u, "o"] \\
& & \emptyset \arrow[r, hook] \arrow[u, "o"] & \bullet \arrow[r, hook] \arrow[u, "o"] & \bullet \arrow[u, "o"] \\
& & & \emptyset \arrow[r, hook] \arrow[u, "o"] & \bullet \arrow[u, "o"] \\
& & & & \emptyset \arrow[u, "o"]
\end{tikzcd}$$

Here we define the morphisms to be morphisms in $\mathsf{Fun}(\tilde{\mathsf{Ar}}[n], \mathsf{C})$ such that the top row squares are cartesian. From this we get that $\tilde{S}_\bullet\mathsf{C}$ is a subtractive category with levelwise fibrations/cofibrations/subtraction sequences, and $[n] \to \tilde{S}_n\mathsf{C} \in \mathsf{sCat}$ is a simplicial category. We can let $\mathsf{K}(\mathsf{C}) = \Omega|w.\tilde{S}_\bullet\mathsf{C}|$. One can check that $\pi_0\mathsf{K}(\mathsf{C}) = \mathsf{K}_0(\mathsf{C})$. Moreover, as $\tilde{S}_\bullet\mathsf{C}$ is itself a subtractive Waldhausen category we can let

$$\mathsf{K}(\mathsf{C})(\ell) = \left|\mathcal{N}(w\tilde{S}_\bullet \cdots \tilde{S}_\bullet\mathsf{C})\right|$$

where the $\tilde{S}_\bullet$ construction is applied ℓ times.

Remark 9.12. Explicitly $\tilde{S}_\bullet^{(\ell)}\mathsf{C} \leq \mathsf{Fun}(\tilde{\mathsf{Ar}}[n_1] \times \cdots \tilde{\mathsf{Ar}}[n_k], \mathsf{C})$ is a subcategory such that each slide is valid for all but one n_i and restrictions to top rows for each $\tilde{\mathsf{Ar}}[n_i]$ are Cartesian. We get sequence of spaces $\mathsf{K}(\mathsf{C})(k)$ for $k \in \mathbb{Z}_{\geq 0}$ which assemble to a symmetric sequence (i.e. we have an S_k action on each $\mathsf{K}(\mathsf{C})(k)$) which assemble to a spectrum.

Theorem 9.13 (Additivity)**.** *The map*

$$\begin{aligned}(s,q) : F_1^+\mathsf{C} &\to \mathsf{C}\times\mathsf{C}\\ X \hookrightarrow Y \twoheadleftarrow Z &\mapsto (X,Z)\end{aligned}$$

induces a homotopy equivalence of simplicial sets $\tilde{S_\bullet}(F_1^+\mathsf{C}) \simeq \tilde{S_\bullet}\mathsf{C}\times\tilde{S_\bullet}\mathsf{C}$.

Remark 9.14 (Delooping)**.** We want $\Omega\mathsf{K}(\mathsf{C})(n) \to \mathsf{K}(\mathsf{C})(n+1)$. For $X_\bullet$ a simplicial object, there is path fibration $PX_\bullet \to X_0$ by taking the constant path, as in the following diagram:

$$\begin{tikzcd} X. \arrow[r] & PX_0 \arrow[d, "\sim"] \\ & X_0 \end{tikzcd}$$

And we also have a diagram

$$\begin{tikzcd} X_1 \arrow[r, equal] \arrow[d, dashed, "="] & X_1 \arrow[r, equal] \arrow[d, dashed, "s_1"] & X_1 \arrow[r, equal] \arrow[d, dashed, "s_1^2"] & \ldots \\ X_1 \arrow[r, dashed, shift right, "s_1"'] \arrow[d, dashed, "d_0"] & X_2 \arrow[l, shift right, "d_1{,}d_2"'] \arrow[r, dotted, "s_1{,}s_2"] \arrow[d, dashed, "d_0"] & X_3 \arrow[r] \arrow[d, dashed, "d_0"] & \ldots \\ X_0 \arrow[r, dashed, "\cdots"] & X_1 \arrow[r, dashed, "\cdots"] & X_2 \arrow[r] & \ldots \end{tikzcd}$$

Applying this to $X_\bullet = w\tilde{S_\bullet}\mathsf{C} \in \mathsf{sCat}$, we get a sequence whose composite is constant and whose middle term is contractible:

$$w\tilde{S_\bullet}\mathsf{C} \to Pw\tilde{S_\bullet}\mathsf{C} \to \mathsf{w.C}.$$

This yields a map $|\mathcal{N}(w\mathsf{C})| \to \Omega\left|\mathcal{N}(w\tilde{S_\bullet}\mathsf{C})\right|$, which is not necessarily a weak equivalence but becomes such after another application of $S_\bullet$.

Remark 9.15. We are interested in lifting the classical setting of maps $\mathsf{K}(\mathsf{Var}_k) \to R$ for R some spectrum. We'll try to lift classical motivic measures to morphisms of spectra $\mathsf{K}(\mathsf{Var}_{/k}) \to R$.

Definition 9.16 (W-exact functor)**.** A W-exact functor $\mathsf{C} \to \mathsf{W}$ from a SW category to a Waldhausen category, is a pair $F_!$ and $F^!$ where

$$\begin{aligned}F_! &: \mathsf{cof}(\mathsf{C}) \to \mathsf{W}\\ F^! &: \mathsf{fib}(\mathsf{C})^{\mathsf{op}} \to \mathsf{W}\end{aligned}$$

where the "op" is to account for our fibrations going the wrong way, such that $F_!(X) = F^!(X)$, base change is satisfied, and subtractive sequences are sent to cofibration sequences.

Such a W-exact functor induces a map $w\tilde{S}_\bullet \mathsf{C} \to wS_\bullet \mathsf{W}$ for simplicial sets and hence a map of spectra $\mathsf{K}(\mathsf{C}) \to \mathsf{K}(\mathsf{W})$. We end with a few illustrative examples:

Example 9.17 (The unit map)**.** Let FinSet_* be the category of pointed finite sets. Then the Barratt-Priddy-Quillen theorem gives that $\mathsf{K}(\mathsf{FinSet}_*) = \mathcal{S}$, so the K-theory of FinSet_* is the sphere spectrum. We want to find a map $\mathcal{S} \to \mathsf{K}(\mathsf{Var}_k)$, and we have the map

$$\begin{aligned} \mathsf{FinSet}_* &\to \mathsf{Var}_k \\ [n] &\mapsto \coprod_{0 \leq i \leq n} \mathsf{Spec}(k) \end{aligned}$$

One can check that this is an op-W functor, hence we get a map $\mathsf{K}(\mathsf{FinSet}_*) \to \mathsf{K}(\mathsf{Var}_k)$ which is the unit map for the E^∞-structure.

Example 9.18 (The point-counting map)**.** We have a point-counting map

$$\begin{aligned} \mathsf{Var}_k &\to \mathsf{FinSet}_* \\ X &\mapsto |X(k)| \end{aligned}$$

where $X(k)$ denotes the k-points of X. We can likewise check this is W-exact and hence we get an induced map on K-theory. Moreover, combined with the previous example, the following diagram commutes:

$$\begin{tikzcd} \mathsf{K}(\mathsf{Var}_k) \arrow[r] & \mathsf{K}(\mathsf{FinSet}_*) \\ \mathsf{K}(\mathsf{FinSet}_*) \arrow[u] \arrow[ur, "\mathrm{id}"'] & \end{tikzcd}$$

Given that $\mathsf{K}(\mathsf{FinSet}_*) = \mathcal{S}$ we see that

$$\mathsf{K}(\mathsf{Var}_k) = \mathcal{S} \oplus \text{"something else"}$$

10 Talk 10: Derived Motivic Measures (Aidan Lindberg)

Reference: [CWZ; BGN21].

10.1 Setup

Remark 10.1. We have a spectrum $\mathsf{K}(\mathcal{V}_k)$; we'll consider maps out of this to detect nontrivial homotopy groups. We have several examples of motivic measures:

Example 10.2. The point measure

$$\begin{aligned} \mathcal{V}_{\mathbb{F}_q} &\to \mathsf{FinSet} \\ X &\mapsto X(\mathbb{F}_q) \end{aligned}$$

where FinSet is the category of finite sets. This induces a map

$$\mathsf{K}_0(\mathcal{V}_{\mathbb{F}_q}) \to \mathsf{K}_0(\mathsf{FinSet}),$$

and one of our goals will be to lift this to a morphism of spectra to make them *derived* motivic measures.

Example 10.3. Let $\mathcal{V}_k^{\times}$ be the category of varieties over k equipped with an automorphism. There is a functor

$$\begin{aligned} \mathcal{V}_{\mathbb{C}}^{\times} &\to \mathsf{ho}(\mathsf{Ch}^b(\mathbb{Q})) \\ X &\mapsto C_c^{\bullet}(X(\mathbb{C}); \mathbb{Q}), \end{aligned}$$

where $X(\mathbb{C})$ is the associated analytic space, $\mathsf{ho}(\mathsf{Ch}^b\mathbb{Q})$ is the homotopy category of bounded chain complexes of $\mathbb{Q}$-modules, and the map is taking compactly supported singular cochains. One can recover the compactly supported Euler characteristic χ as a map on K_0 sending X to $\chi(X) := \sum_{i\in\mathbb{Z}} \dim H^i(X(\mathbb{C}); \mathbb{Q})$.

Example 10.4. One can take the action of Frobenius on ℓ-adic cohomology to obtain a functor

$$\begin{aligned} \mathcal{V}_{\mathbb{F}_q}^{\times} &\to \mathsf{hoCh}^b(\mathrm{Aut}(\mathbb{Q}_\ell)) \\ X &\mapsto \mathrm{Frob}_q \curvearrowright \mathbb{R}\Gamma_c(X_{\bar{\mathbb{F}}_q}; \mathbb{Q}_\ell), \end{aligned}$$

which is an ℓ-adic version of the Euler characteristic.[17] Applying K_0 gives an associated zeta function.

Remark 10.5. There is a category Mot_k which any "nice" (additive) invariant of $\mathcal{V}_k$ should factor through. This is a dg-category, and taking the perfect objects and applying the Waldhausen construction yields a K-theory. Note that existence of Mot_k is partially conjectural, although we do have its derived category.

[17] This can be thought of as giving continuous Galois representations.

Remark 10.6. For C a Waldhausen category we can associate the spectrum $\mathsf{K}(\mathsf{C})$. Note that $\mathcal{V}_k$ is a SW category, so the arrows go the wrong way, and thus we want a map $\mathsf{K}(\mathsf{C}_{\mathsf{SW}}) \to \mathsf{K}(\mathsf{C}_{\mathcal{W}})$.

10.2 W-exact functors

Definition 10.7 (W-exact functors)**.** Let C be an SW category and D be a Waldhausen category. Then a **weakly W-exact functor** $F : \mathsf{C} \to \mathsf{D}$ is a triple $F = (F_!, F^!, F^w)$ where

1. $F_! : \mathsf{cof}(\mathsf{C}) \to \mathsf{D}$, where $\mathsf{cof}(-)$ is the subcategory generated by cofibrations,
2. $F^! : \mathsf{fib}(\mathsf{C})^{\mathrm{op}}$, where $\mathsf{fib}(-)$ are "complement" maps $\begin{tikzcd} Z \arrow[r, "\circ" marking] & Y \end{tikzcd}$ in subtraction sequences
$$\begin{tikzcd} X \arrow[r, hook, "i"] & Y & Z \arrow[l, "\circ" marking] \end{tikzcd} ,$$
3. $F^w : W(\mathsf{C}) \to W(\mathsf{D})$ where $W(-)$ denotes the weak equivalences,
4. $F_! = F^! = F^w$ on all objects.
5. Certain cartesian squares are sent to cartesian squares:
$$\begin{tikzcd} X \arrow[r, hook, "j"] \arrow[d, "\circ" marking, "i"'] & Z \arrow[d, "\circ" marking, "i^!"'] & & F(X) \arrow[r, "j_!"] & F(Z) \\ Y \arrow[r, hook, "j^!"] & W \arrow[r, Rightarrow, "F"] & {} & F(Y) \arrow[u, "i^!"] \arrow[r, "\tilde{j}_!"] & F(W) \arrow[u, "(i')^!"'] \end{tikzcd}$$
6. Subtraction sequences $\begin{tikzcd} X \arrow[r, hook, "i"] & Y & Y \setminus X \arrow[l, "\circ" marking, "j"'] \end{tikzcd}$ are sent to cofiber sequences:
$$\begin{tikzcd} F(X) \arrow[r, "i_!"] \arrow[d] & F(Y) \arrow[d, "j^!"] \\ F(\emptyset) \arrow[r] & F(Y - X) \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

Proposition 10.1. *W-exact functors induces maps* $\mathsf{K}(\mathsf{C}) \to \mathsf{K}(\mathsf{D})$.

Remark 10.8. Goal: construct W-exact functors $\mathcal{V}_k \to \mathsf{C}$ for appropriate C.

Example 10.9. Let $\mathcal{V}_k^{\mathrm{cpt}}$ be the SW category whose objects are open embeddings $\begin{tikzcd} X \arrow[r, "\circ" marking] & \overline{X} \end{tikzcd}$ with $\overline{X}$ proper and morphisms are commuting squares involving $f, \overline{f}$. Morphisms are cofibrations if $f, \overline{f}$ are closed embeddings, and complements if f is an open embedding and $\overline{f}$ is a closed embedding, and take weak equivalences to be isomorphisms. A sequence

$$(Z \to \overline{Z}) \hookrightarrow (X \to \overline{X}) \leftarrow\!\circ\!- (U \to \overline{U})$$

is a subtraction sequence if $Z \hookrightarrow X \leftarrow\!\circ\!- U$ is subtraction sequence in $\mathcal{V}_k$ and $\operatorname{im}(\overline{U} \to \overline{X}) = \overline{(\overline{X} \setminus \overline{Z})}$, and this makes $\mathcal{V}_k^{\mathrm{cpt}}$ a SW category. Note that there is a forgetful map $\mathcal{V}_k^{\mathrm{cpt}} \to \mathcal{V}_k$ where $(X \to \overline{X}) \mapsto X$.

Warnings 10.10. A functor to *just* the homotopy (or derived) category is insufficient to induce a map on K-theory!

Theorem 10.11. *Let $k \hookrightarrow \mathbb{C}$ be a subfield and let $\mathsf{Ch}^b(R)$ for R a commutative ring be the category of bounded chain complexes which are homologically finite. Let $[C_c(-;R)]$ denote the class of R-valued singular cochains in the homotopy category $\mathsf{hoCh}^b(R)$. Then the functor*

$$\begin{aligned} \mathcal{V}_k &\to \mathsf{Ch}^b(R)^{\mathrm{op}} \\ X &\mapsto [C_c(X;R)] \end{aligned}$$

admits a model as a span of weakly W-exact functors:

$$\begin{tikzcd} & \mathcal{V}_k^{\mathrm{cpt}} \arrow[dl, "U,\sim"'] \arrow[dr, "G"] & \\ \mathcal{V}_k & & \mathsf{Ch}^b(R)^{\mathrm{op}} \end{tikzcd}$$

Note that G will be defined in the proof, and this produces a factorization of the functor above.

Proof.

- Define $G(X,\overline{X}) \coloneqq C_{\mathrm{sing}}(X, \overline{X} \setminus X)$
- Note

$$\begin{aligned} G_! : \mathsf{fib}(\mathcal{V}^{\mathrm{cpt}}) &\to \mathsf{Ch}^b(R)^{\mathrm{op}} \\ \left(Z, \left(\overline{Z}\right) \xrightarrow{\phi} (X, \overline{X})\right) &\mapsto \phi^*. \end{aligned}$$

  where ϕ is a closed embedding, and

$$\begin{aligned} G^! : \mathrm{cofib}(\mathcal{V}^{\mathrm{cpt}}) &\to \mathsf{Ch}^b(R)^{\mathrm{op}} \\ \left((Z, \overline{Z}) \xrightarrow{\psi} (X, \overline{X})\right) &\mapsto \text{Extension by zero} \end{aligned}$$

  where ψ is an open embedding.

- Check that U induces an equivalence on K-theory, see [**CWZF19**][Lemma 2.21] on derived ℓ-adic zeta functions).[18]

$\square$

Theorem 10.12. *If k is a subfield of $\mathbb{C}$ then $\mathsf{K}_i(\mathcal{V}_k) \neq 0$ for infinitely many i. In particular $\mathsf{K}_{4s-1}(\mathcal{V}_k) \neq 0$ for all positive s.*

Brief sketch of proof. Use the W-exact functor of the previous theorem to get a nonzero map $\mathsf{K}_{4s-1}(\mathcal{V}_k) \to \mathsf{K}_{4s-1}(\mathbb{Z})$ whose image is nonzero.

$\square$

Question 10.13. It has been shown that $\mathsf{K}(\mathcal{V}_k) = \mathcal{S} \oplus \tilde{\mathsf{K}}(\mathcal{V}_k)$ and $\mathcal{S} = \mathsf{K}(\mathsf{FinSet})$. Do there exist classes that do not come from the image of $\mathcal{S}$ under the $\mathbb{E}_\infty$ ring structure?[19]

Remark 10.14. [BGN21] constructs classes of infinite order in $\mathsf{K}(\mathcal{V}_k)$.

[18] In fact, U is exact.

[19] Why $4s - 1$? This has something to do with the orthogonal group and the J-homomorphism.

11 Talk 11: Higher Algebraic K-theory (Danika Van Niel)

Reference: [Wei13, p. IV.7] and [Wal85, §1.3 and 1.9].

Our exposition will closely follow the history of K-theory.

Lower algebraic K-theory groups	rings A	$\mathsf{K}_0, \mathsf{K}_1, \mathsf{K}_2$ (1957–1967)	
Higher algebraic K-theory	rings A	$+$-construction (Quillen, 1971 [Qui71])	
	small exact categories $\mathcal{E}$	Q-construction (Quillen, 1972 [Qui73])	e.g. schemes
	Waldhausen categories $\mathcal{C}$	$S_\bullet$-construction (Waldhausen, 1978 [Wal85])	e.g. Top

Definition 11.1 (K_0, due to Grothendieck)**.** Let A be an associative, unital ring and $\mathcal{P}(A)$ the monoid (under $\oplus$) of isomorphism classes of finitely generated projective A-modules. Define

$$\mathsf{K}_0(A) \coloneqq \mathrm{Gr}(\mathcal{P}(A)),$$

where Gr denotes the Grothendieck group, i.e. the group completion of a monoid.

Example 11.2. If F is a field, then $\mathcal{P}(F) \cong \mathbb{N}$ since vector spaces up to isomorphism are classified by dimension, and thus $\mathsf{K}_0(F) \cong \mathbb{Z}$.

Definition 11.3 (K_1, due to Bass-Schanuel)**.** For A an associative unital ring, define

$$\mathsf{K}_1(A) \coloneqq \mathrm{GL}(A)/E(A),$$

where $\mathrm{GL} \coloneqq \mathrm{colim}_n \, \mathrm{GL}_n(A)$ and $E(A)$ is generated by certain elementary matrices.

Remark 11.4. The next K-group $\mathsf{K}_2(A)$ was defined by Milnor, but we will not go into the details here. The point is that higher K-theory should generalize these existing definitions.

11.1 Quillen's $+$-construction and Q-construction

Remark 11.5. We first start with the $+$-construction. We want a space $\mathsf{K}(A)$ such that $\pi_1 \mathsf{K}(A) \cong \mathsf{K}_1(A)$. We know $\pi_1 \mathrm{BGL}(A) \cong \mathrm{GL}(A)$, so to get the quotient $\mathsf{K}_1(A)$ we glue in appropriate cells so that the fundamental group of $\mathsf{K}(A)$ is what we want and

$$H_*(\mathsf{K}(A), M) \cong H_*(\mathrm{BGL}(A), M)$$

for any $\mathbb{Z}[\mathsf{K}_1(A)]$-module M. Quillen does this using the +-construction to get a space $\mathrm{BGL}(A)^+$ and defines

$$\mathsf{K}(A) \coloneqq \mathrm{BGL}(A)^+, \qquad \mathsf{K}_i(A) \coloneqq \pi_i \, \mathrm{BGL}(A)^+$$

for $i > 0$.

After defining the ±construction, Quillen defined a more generalized construction.

Remark 11.6. Recall that a **small** category $\mathcal{C}$ is a category such that $\mathrm{Ob}\,\mathcal{C}$ and $\mathrm{Hom}\mathcal{C}$ are set-sized. An **exact** category is a pair $(\mathcal{E}, S)$ where $\mathcal{E}$ is an additive category[20] and S is a family of sequences in $\mathcal{E}$ of the form

$$0 \to M' \rightarrowtail M \twoheadrightarrow M'' \to 0 \tag{$*$}$$

such that $\mathcal{E}$ is a full subcategory of some abelian category $\mathcal{A}$, plus some conditions. Recall this means $\mathcal{A}$ has kernels and cokernels, monomorphisms are kernels, and epimorphisms are cokernels; fullness of $\mathcal{E}$ in $\mathcal{A}$ means $\mathrm{Ob}\,\mathcal{E} \subseteq \mathrm{Ob}\,\mathcal{A}$ and for all objects X, Y, $\mathcal{E}(X, Y) = \mathcal{A}(X, Y)$. The idea of an exact category is that we think of $(*)$ as an exact sequence in $\mathcal{E}$. In $(*)$, the arrows $\rightarrowtail$ are called **admissible monomorphisms** and the arrows $\twoheadrightarrow$ are called **admissible epimorphisms**.

Example 11.7. For a ring A, $\mathrm{iso}\,\mathcal{P}(A)$ is a small exact category where the objects are the same and the only morphisms are isomorphisms.

Remark 11.8. Let $\mathcal{E}$ be a small, exact category. Define the generators of $K_0(\mathcal{E})$ to be isomorphism classes of objects in $\mathcal{E}$ with the relation $[M] = [M'] + [M'']$ for all sequences $(*)$. To form the higher K-groups, we define a new category $Q\mathcal{E}$. The objects of $Q\mathcal{E}$ are the objects of $\mathcal{E}$, but a morphism from X to Y in $Q\mathcal{E}$ is a span[21]

$$X \twoheadleftarrow M \rightarrowtail Y.$$

Definition 11.9 (Quillen Q-construction)**.** For $\mathcal{E}$ a small, exact category, define

$$\mathsf{K}(\mathcal{E}) \coloneqq \Omega \mathrm{B} Q\mathcal{E}, \qquad \mathsf{K}_n(\mathcal{E}) \coloneqq \pi_{n+1} \mathrm{B} Q\mathcal{E} \text{ for } n \geq 0.$$

Theorem 11.10 $(+ = Q)$**.** *For A a unital, associative ring,*

$$K_n^+(A) \cong K_n^Q(P\,\mathrm{iso}\,\mathcal{P}(A)),$$

where K^+ indicates the use of the +-construction and K^Q indicates the use of the Q-construction.

[20] This means there is a 0 element, a map $\mathcal{E} \times \mathcal{E} \to \mathcal{E}$, and for all objects X, Y the hom set $\mathcal{E}(X, Y)$ is an abelian group.

[21] Really, an equivalence class of them, where the middle object is given only up to isomorphism.

Remark 11.11. To prove this, Quillen uses another construction called the $S^{-1}S$ construction for symmetric monoidal categories. Quillen shows the following:

- For $S := \operatorname{iso}\mathcal{P}(A)$, one has $\Omega \mathrm{B}Q\mathcal{P}(A) \simeq \mathrm{B}S^{-1}S$, and
- $\mathrm{B}S^{-1}S \simeq \mathbb{Z} \times \mathrm{BGL}(A)^+$.

Thus

$$\pi_n(\Omega \mathrm{B}Q(\mathcal{P}(A))) \cong \pi_n(\mathrm{B}S^{-1}S) \cong \pi_n(\mathrm{BGL}(A)^+)$$

for all $n > 0$, which relates the Q-construction to the $+$-construction.

11.2 Waldhausen's $S_\bullet$-construction

We have now seen that the $+$ and Q constructions agree where they overlap. A few years later, Waldhausen defined an even more general construction.

Remark 11.12. Waldhausen uses the $S_\bullet$-construction to define higher K-theory for Waldhausen categories, which are categories with cofibrations ($\rightarrowtail$) and weak equivalences ($\xrightarrow{\sim}$). In a Waldhausen category, we have access to *cofibration sequences*

$$A \rightarrowtail B \twoheadrightarrow B/A := \operatorname{coker}(A \rightarrowtail B).$$

The $S_\bullet$-construction builds a simplicial Waldhausen category

$$S_\bullet\mathcal{C} = \{S_0\mathcal{C} \;\substack{\longleftarrow \\ \longrightarrow \\ \longleftarrow}\; S_1\mathcal{C} \;\substack{\longleftarrow \\ \longrightarrow \\ \longleftarrow \\ \longrightarrow \\ \longleftarrow}\; S_2\mathcal{C} \cdots\}$$

where each $S_i\mathcal{C}$ is a Waldhausen category. Then the nerve of this simplicial category is a bisimplicial set, and we restrict to the bisimplices whose vertical morphisms are all weak equivalences. The realization of the resulting object is denoted $|wS_\bullet\mathcal{C}|$ and we define

$$\mathsf{K}(\mathcal{C}) = \Omega\,|wS_\bullet\mathcal{C}| \quad \text{and} \quad \mathsf{K}_n(\mathcal{C}) = \pi_{n+1}(|wS_\bullet\mathcal{C}|).$$

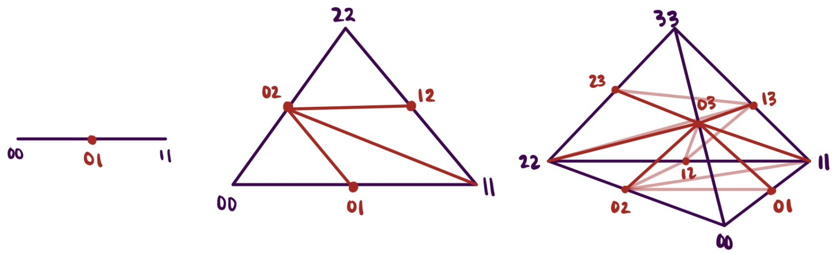
Like the "$+ = Q$" theorem, there is an "$S_\bullet = Q$" theorem. Every small exact category $\mathcal{E}$ can be viewed as a Waldhausen category whose cofibrations are the admissible monomorphisms and whose weak equivalences are the isomorphisms. To emphasize this second point, we will use $iS_\bullet\mathcal{E}$ to denote $wS_\bullet\mathcal{E}$.

A key tool in comparing $iS_\bullet\mathcal{E}$ to the Q-construction is the following:

Definition 11.13 (Segal's edgewise subdivision)**.** The **edgewise subdivision** of a simplicial set $X_\bullet$ is another simplicial set $X^e_\bullet$ with $X^e_n = X_{2n+1}$. The map $[n] \mapsto [2n+1]$ is given by sending

$$0 < 1 < \cdots < n \qquad \text{to} \qquad n' < \cdots < 1' < 0 < 1 < \cdots < n.$$

Geometrically, for small n it looks like the following:



Theorem 11.14. *For $X_\bullet$ a simplicial set, there is a natural homeomorphism*

$$|X_\bullet| \cong |X_\bullet^e|.$$

Remark 11.15. We want to compare $S_\bullet^e\mathcal{E}$ to $Q_\bullet E$, where $Q_n\mathcal{E} = N_nQ\mathcal{E}$. Let $s_\bullet\mathcal{E}$ denote the simplicial set of objects of $S_\bullet\mathcal{E}$. Seeing $s_\bullet\mathcal{E}$ as a simplicial category in the trivial way, we have maps

$$s_k^e\mathcal{E} = s_{2k+1}\mathcal{E} \to Q_k\mathcal{E}.$$

Let's see how this works for small k.

For $k = 0$:

$$\begin{aligned} s_0^e\mathcal{E} = s_1\mathcal{E} &\longrightarrow Q_0\mathcal{E} \\ A_1 &\longmapsto A_1, \end{aligned}$$

For $k = 1$:

$$s_1^e\mathcal{E} = s_3\mathcal{E} \longrightarrow Q_1\mathcal{E}$$

$$\begin{tikzcd}
 & & A_{23} & & & & \\
 & A_{12} \arrow[r, tail] & A_{13} \arrow[u, two heads] & \longmapsto & & A_2 \arrow[dl, two heads] \arrow[dr, tail] & \\
A_1 \arrow[r, tail] & A_2 \arrow[r, tail] \arrow[u, two heads] & A_3 \arrow[u, two heads] & & A_{12} & & A_3
\end{tikzcd}$$

For $k = 2$:

$$s_3^e\mathcal{E} = s_5\mathcal{E} \longrightarrow Q_2\mathcal{E}$$

$$\begin{tikzcd}
 & & & & A_{45} & & & & & & \\
 & & & A_{34} \arrow[r, tail] & A_{35} \arrow[u, two heads] & & & & & & \\
 & & A_{23} \arrow[r, tail] & A_{24} \arrow[r, tail] \arrow[u, two heads] & A_{25} \arrow[u, two heads] & \longmapsto & & & A_3 \arrow[dl, two heads] \arrow[dr, tail] & & \\
 & A_{12} \arrow[r, tail] & A_{13} \arrow[r, tail] \arrow[u, two heads] & A_{14} \arrow[r, tail] \arrow[u, two heads] & A_{15} \arrow[u, two heads] & & & A_{13} \arrow[dl, two heads] \arrow[dr, tail] & & A_4 \arrow[dl, two heads] \arrow[dr, tail] & \\
A_1 \arrow[r, tail] & A_2 \arrow[r, tail] \arrow[u, two heads] & A_3 \arrow[r, tail] \arrow[u, two heads] & A_4 \arrow[r, tail] \arrow[u, two heads] & A_5 \arrow[u, two heads] & & A_{23} & & A_{14} & & A_5
\end{tikzcd}$$

In fact, $s^e_\bullet\mathcal{E} \to Q_\bullet\mathcal{E}$ is simplicial and surjective. We can extend this map $iS^e_\bullet\mathcal{E} \to iQ_\bullet\mathcal{E}$ which is a levelwise equivalence of categories, and thus geometrically realizes to a homotopy equivalence. Finally, we use the "swallowing lemma" to show $|iQ_\bullet\mathcal{E}| \simeq \mathrm{B}Q\mathcal{E}$. Thus

$$|iS_\bullet\mathcal{E}| \simeq |iS^e_\bullet\mathcal{E}| \simeq |iQ_\bullet\mathcal{E}| \simeq \mathrm{B}Q\mathcal{E}$$

and so $\pi_{n+1}(|iS_\bullet\mathcal{E}|) \cong \pi_{n+1}(\mathrm{B}Q\mathcal{E})$, which shows the two constructions agree.

12 Talk 12 : CGW-categories (Chloe Lewis)

Reference: [CZ18].

Remark 12.1. Recall from Talk 0 that one of the key observations for building K-theory was that there were two ways for an object to be smaller than another, which we indicated using different colored arrows:

$$P \hookrightarrow Q \quad \text{and} \quad P \hookrightarrow Q.$$

a In the context of trying to understand K-theory of varieties, we have things like

$$X \hookrightarrow Y \hookleftarrow Y \setminus X,$$

but the second arrow is going the wrong way. So how can we do K-theory here?

The idea is to generalize exact categories to **CGW categories** and develop a K-theory in this context so that our favorite K-theory theorems and constructions still work. Before Campbell-Zakharevich worked on CGW categories and their K-theory, they separately developed different ways of building the K-theory of varieties (using subtractive categories and assemblers, respectively).

Upshot 12.2. We can use CGW categories to show that the K-theories of Campbell and Zakharevich coincide for $\mathcal{V}_k$ the category of varieties over a field k:

$$\mathsf{K}^C(\mathcal{V}_k) \simeq \mathsf{K}^Z(\mathcal{V}_k).$$

12.1 CGW categories

Definition 12.3 (Double categories)**.** A **double category** $\mathcal{C}$ is two categories $\mathcal{M}$ and $\mathcal{E}$ with the same objects, $\mathrm{Ob}\,\mathcal{M} = \mathrm{Ob}\,\mathcal{E}$. We denote the morphisms of $\mathcal{M}$ as $A \rightarrowtail B$ and the morphisms of $\mathcal{E}$ as $A \circ\!\!\longrightarrow B$. The double category $\mathcal{C}$ also comes with distinguished squares

$$\begin{tikzcd} A \arrow[r, tail] \arrow[d, "\circ" marking] \arrow[dr, phantom, "\square"] & B \arrow[d, "\circ" marking] \\ C \arrow[r, tail] & D \end{tikzcd}$$

Definition 12.4 (CGW categories)**.** A **CGW category** is a double category $\mathcal{C}$ along with

- An isomorphism of categories $\phi\colon \operatorname{iso}\mathcal{M} \to \operatorname{iso}\mathcal{E}$,
- Two equivalences of categories, $c\colon \mathsf{Ar}_\square\mathcal{M} \to \mathsf{Ar}_\triangle\mathcal{E}$ and $k\colon \mathsf{Ar}_\square\mathcal{E} \to \mathsf{Ar}_\triangle\mathcal{M}$. Here $\mathsf{Ar}_\square M$ is the category whose objects are arrows of $\mathcal{M}$, $A \circ\!\!\longrightarrow B$, and whose morphisms are distinguished squares. The category $\mathsf{Ar}_\triangle\mathcal{E}$ is

slightly different, with objects $A \to B$ and morphisms commuting squares in $\mathcal{E}$ whose top horizontal arrow is an isomorphism,

$$\begin{tikzcd} & A \cong A' \arrow[dl, circ] \arrow[dr, circ] & \\ C \arrow[rr, circ] & & D \end{tikzcd}.$$

The categories $\mathsf{Ar}_\square\mathcal{E}$ and $\mathsf{Ar}_\triangle\mathcal{M}$ are formed analogously. We use k and c to build kernels and cokernels.

These must satisfy the following axioms:

Z) *Zero:* there is an object $\emptyset$ which is initial in both $\mathcal{M}$ and $\mathcal{E}$,

I) *Isomorphisms:* for any isomorphism $f : A \circ\!\!\longrightarrow B$, we have distinguished squares

$$\begin{tikzcd} A \arrow[r, tail, "f"] \arrow[d, circ, "\phi(f)"'] \arrow[dr, phantom, "\square"] & B \arrow[d, circ, "id"] \\ C \arrow[r, tail, "id"'] & D \end{tikzcd} \quad \text{and etc.}$$

M) *Monics:* all morphisms in $\mathcal{M}$ and $\mathcal{E}$ are monic,

K) *Kernels:* for all $g : A \circ\!\!\longrightarrow B$, write $k(g) = g^k : A^{g/k} \rightarrowtail B$. Then the square

$$\begin{tikzcd} \emptyset \arrow[r, tail] \arrow[d, circ] \arrow[dr, phantom, "\square"] & A \arrow[d, circ, "g"] \\ A^{g/k} \arrow[r, tail, "g^k"'] & B \end{tikzcd}$$

is distinguished.

C) *Cokernels:* dual statement as above for c,

A) K_0 *is abelian:* for all objects A, B there is an object X such that

$$\begin{tikzcd} \emptyset \arrow[r, tail] \arrow[d, circ] \arrow[dr, phantom, "\square"] & A \arrow[d, circ] \\ B \arrow[r, tail] & X \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \emptyset \arrow[r, tail] \arrow[d, circ] \arrow[dr, phantom, "\square"] & B \arrow[d, circ] \\ A \arrow[r, tail] & X \end{tikzcd}.$$

We think of X as "$A \oplus B$".

Example 12.5. Any exact category is CGW, where we set $\mathcal{M}$ to be admissible monomorphisms and $\mathcal{E}$ to be (admissible epimorphisms)$^{\mathrm{op}}$. The distinguished squares are stable squares (squares which are both pushouts and pullbacks). If $A \to B$ is an epimorphism, then we get a morphism $B \circ\!\!\longrightarrow A$ in $\mathcal{E}$ and we define k to send this to the monomorphism

$$\ker(A \to B) \rightarrowtail A \ .$$

Similarly, if $C \rightarrowtail D$ is in $\mathcal{M}$ then define c to take it to

$$\operatorname{coker}(C \rightarrowtail D) \circ\!\!\longrightarrow D \ ,$$

the opposite of the epimorphism $D \to \operatorname{coker}(C \rightarrowtail D)$.

Example 12.6. Finite sets are CGW but not exact – note that $\emptyset$ is initial but not terminal. In this case, $\mathcal{M} = \mathcal{E}$ are the inclusions and $\square$s are stable squares. Both c and k send an inclusion $A \hookrightarrow B$ to the inclusion $B \setminus A \hookrightarrow B$.

Example 12.7. Varieties over k (or schemes of finite type) are CGW. Here, $\mathcal{M}$ is closed immersions, $\mathcal{E}$ is open immersions, and $\square$s are pullbacks:

$$\begin{tikzcd} X \arrow[r, tail, "f"] \arrow[d, "g"', circ] & Y \arrow[d, circ] \\ W \arrow[r, tail] & Z = \operatorname{im}(f) \cup \operatorname{im}(g) \end{tikzcd}$$

12.2 K-theory of CGW categories

Lemma 12.8. *A span* $A \xrightarrow{f} B \circ\!\!\xrightarrow{g} C$ *gives a square*

$$\begin{tikzcd} A \arrow[r, tail, "f"] \arrow[d, circ] \arrow[dr, phantom, "\square"] & B \arrow[d, circ, "g"] \\ D \arrow[r, tail] & C \end{tikzcd} .$$

Remark 12.9. An important theorem is that we can do the Q-construction on CGW categories. For a CGW category $\mathcal{C}$, the objects of $Q\mathcal{C}$ are the objects of $\mathcal{C}$ and morphisms are spans. We define

$$\mathsf{K}(\mathcal{C}) = \Omega BQ\mathcal{C}.$$

We can check that $\mathsf{K}_0(\mathcal{C}) \cong \mathbb{Z}[\operatorname{Ob}\mathcal{C}]/\sim$ where $\sim$ indicates $[D]+[B] = [A]+[C]$ whenever there is a square

$$\begin{tikzcd} A \arrow[r, tail, "f"] \arrow[d, circ] \arrow[dr, phantom, "\square"] & B \arrow[d, circ, "g"] \\ D \arrow[r, tail] & C \end{tikzcd}$$

in $\mathcal{C}$. Recall the following:

Theorem 12.10 (Quillen's dévissage for abelian categories)**.** *If an inclusion of abelian categories* $\mathcal{A} \hookrightarrow \mathcal{B}$ *is "nice" and for all* $B \in \mathcal{B}$ *there is a filtration*

$$\emptyset = B_0 \subseteq B_1 \subseteq \cdots \subseteq B_n = B$$

such that $B_i/B_{i-1} \in \mathcal{A}$*, then* $\mathsf{K}(\mathcal{A}) \simeq \mathsf{K}(\mathcal{B})$*.*

Remark 12.11. We built CGW categories in analogy to exact categories, and there is a similar analogy for abelian categories, called *ACGW*, which are like CGW categories with "pushouts." There is also something called **pre-ACGW**, and the theorem is that dévissage holds for (pre-)ACGW categories.

Theorem 12.12 (Dévissage 2.0)**.** *Suppose $\mathcal{A} \hookrightarrow \mathcal{B}$ is a "nice" inclusion of (pre-)ACGW categories and for all $B \in \mathcal{B}$ there is a filtration*

$$\emptyset = B_0 \rightarrowtail B_1 \rightarrowtail \cdots \rightarrowtail B_n = B$$

such that $B_i^{C/B_{i-1}} \in \mathcal{A}$. Here $B_i^{C/B_{i-1}}$ is the object fitting into the square

$$\begin{tikzcd}
\emptyset \arrow[r, tail] \arrow[d, "\circ" marking] \arrow[dr, phantom, "\square"] & B_i^{C/B_{i-1}} \arrow[d, "\circ" marking] \\
B_{i-1} \arrow[r, tail] & B_i
\end{tikzcd}$$

which is given by the lemma. Then $\mathsf{K}(\mathcal{A}) \simeq \mathsf{K}(\mathcal{B})$.

Remark 12.13. For example, $\mathcal{V}_k$ is pre-ACGW and $\mathsf{Sch}^{\mathrm{ft}}$ (the category of schemes of finite type) is ACGW. From the dual statement of dévissage 2.0, we get $\mathsf{K}(\mathcal{V}_k) \simeq \mathsf{K}(\mathsf{Sch}^{\mathrm{ft}})$. We can also do the $S_\bullet$-construction on CGW categories, where $S_n\mathcal{C}$ consists of

$$\begin{tikzcd}
\emptyset \arrow[r, tail] & C_{01} \arrow[r, tail] & \dots \arrow[r, tail] & C_{0n} \\
 & \emptyset \arrow[u, "\circ" marking] \arrow[r, tail] & \dots \arrow[r, tail] & C_{1n} \arrow[u, "\circ" marking] \\
 & & \ddots & \vdots \arrow[u, "\circ" marking] \\
 & & & C_{nn} \arrow[u, "\circ" marking]
\end{tikzcd}$$

and $\mathsf{K}(\mathcal{C}) \simeq \Omega\,|S_\bullet\mathcal{C}|$. Similarly, we can also do the scissors-Waldhausen construction from Talk 9; $\mathsf{Sch}^{\mathrm{ft}}$ is both ACGW and scissors-Waldhausen.

12.3 Comparing Campbell's and Zakharevich's K-theory of varieties

Remark 12.14. The big theorem is that the K-theory of varieties built using Inna's assemblers coincides with the K-theory of varieties built with Jonathan's version of the $S_\bullet$-construction.

Theorem 12.15.

$$\mathsf{K}^Z(\mathcal{V}_k) \simeq \mathsf{K}^C(\mathcal{V}_k).$$

Sketch of proof. The idea of the proof is to use helper CGW categories:

$$\mathsf{K}^C(\mathsf{Var}_{/k}) \xrightarrow{\sim} \mathsf{K}^C(\mathsf{Sch}^{\mathrm{ft}}) \xrightarrow{\sim} \mathsf{K}^C(\mathcal{C}) \xleftarrow{\sim} \mathsf{K}^Z(\mathsf{Var}_{/k}),$$

where the first equivalence follows from dévissage and the second equivalence is difficult. The category $\mathcal{C}$ is a helper category $\mathsf{Sch}^{\mathrm{ft}}W$. For the difficult arrow, the idea is to filter by dimension,

$$\mathsf{K}^C(\mathsf{Sch}^{\mathrm{ft}}) = \operatorname*{hocolim}_{n\geq 0} \mathsf{K}^C(\mathsf{Sch}^{\mathrm{ft},n}),$$

and to look at (somewhat large) diagrams of categories involving more helper categories. □

Upshot 12.16. The main takeaway is that we don't *need* exact categories to do K-theory – one just needs two notions of what it means to be an inclusion and the standard tools and machinery of K-theory still hold.

13 Talk 13: FCGW-categories (Lucy Grossman)

Reference: [SS21].

13.1 (-)CGW Categories

We have been introduced to the concept of a double category in Talk 12, and in this talk we will introduce a variation on this theme. The following notion will be especially useful in what is to follow:

Definition 13.1 (Double isomorphisms)**.** Let $\mathcal{C} = (\mathcal{M}, \mathcal{E})$ be a double category. We say $\mathcal{C}$ has **double isomorphisms** if the following are present:

- A groupoid of functors I

  $$\mathcal{M} \longleftarrow I \longrightarrow \mathcal{E}$$

  providing isomorphisms.

- Given a commuting square

  $$\begin{tikzcd} \cdot \arrow[r, tail, "g"] \arrow[d, tail, "f"] & \cdot \arrow[d, tail, "f'"] \\ \cdot \arrow[r, tail, "g'"] & \cdot \end{tikzcd}$$

  there exists a unique square

  $$\begin{tikzcd} \cdot \arrow[r, tail, "g"] \arrow[d, two heads, "f"] & \cdot \arrow[d, two heads, "f'"] \\ \cdot \arrow[r, tail, "g'"] & \cdot \end{tikzcd}$$

  and vice versa where f denotes both the $\mathcal{M}$-morphism and the $\mathcal{E}$-morphism from the previous criterion.

- The analogous condition for $\mathcal{E}$-morphisms and $\mathcal{E}$--squares.

Remark 13.2. Recall that double categories come with squares relating $\mathcal{E}$--morphism to $\mathcal{M}$-morphisms. In this talk, such squares will be called **pseudo-commutative** or **mixed** squares. These provide one arrow category of interest, but two others are integral to the the definition of the variation of CGW categories we will define:

Definition 13.3 (The triangle arrow category)**.** For $\mathcal{C} = (\mathcal{M}, \mathcal{E})$ a double category, denote by $\mathsf{Ar}_\triangle\mathcal{M}$ the category with

- Objects: The morphisms of $\mathcal{M}$.

- Morphisms: For $A \overset{f}{\rightarrowtail} B$ and $A' \overset{f'}{\rightarrowtail} B'$ objects in $\mathsf{Ar}_\triangle \mathcal{M}$, take $\mathrm{Hom}_{\mathsf{Ar}_\triangle \mathcal{M}}$ to be commutative squares of the form

$$\begin{tikzcd} A \arrow[r, tail, "f"] \arrow[d, tail] & B \arrow[d, tail] \\ A' \arrow[r, tail, "f'"] & B' \end{tikzcd}$$

One likewise defines $\mathsf{Ar}_\triangle \mathcal{E}$. The next definition generalizes this to arbitrary categories:

Definition 13.4 (Good squares)**.** Let $\mathcal{A}$ be a category. A **class of good squares**, denoted $\mathsf{Ar}_g\mathcal{A}$, is the following subcategory:

- Objects: The morphisms of $\mathcal{A}$.
- Morphisms: Commuting squares of morphisms in $\mathcal{A}$, which are depicted by diagrams of the form

$$\begin{tikzcd} \cdot \arrow[r] \arrow[d] \arrow[dr, phantom, "g"] & \cdot \arrow[d] \\ \cdot \arrow[r] & \cdot \end{tikzcd}$$

  Each of these morphisms is called a **good square**.

Remark 13.5. Note that this definition generalizes Definition 13.3, since we now use all morphisms in $\mathcal{A}$ in our squares rather than just those in a particular subcategory of the double category. We are now prepared to define a version of CGW categories equipped with good squares:

Definition 13.6 (g-CGW categories)**.** A **g-CGW** category is a double category $\mathcal{C} = (\mathcal{M}, \mathcal{E})$ along with the following:

- A class of good squares $\mathsf{Ar}_g\mathcal{M}$ in $\mathcal{M}$ and a class of good squares $\mathsf{Ar}_g\mathcal{E}$ in $\mathcal{E}$.
- Equivalences of categories

$$k : \mathsf{Ar}_\circlearrowleft \mathcal{E} \to \mathsf{Ar}_g \mathcal{M}$$

  and

$$c : \mathsf{Ar}_\circlearrowleft \mathcal{M} \to \mathsf{Ar}_g \mathcal{E}$$

such that the following properties hold:
(Z) $\mathcal{M}$ and $\mathcal{E}$ have initial objects, and those initial objects agree.
(M) All morphisms in $\mathcal{M}$ or $\mathcal{E}$ are monic.
(G) Both $\mathsf{Ar}_\triangle\mathcal{M} \subseteq \mathsf{Ar}_g\mathcal{M} \subseteq \mathsf{Ar}_x\mathcal{M}$ and $\mathsf{Ar}_\triangle\mathcal{E} \subseteq \mathsf{Ar}_g\mathcal{E} \subseteq \mathsf{Ar}_x\mathcal{E}$ where $\mathsf{Ar}_x\mathcal{A}$ stands for the pullback squares in any category $\mathcal{A}$.
(D) k sends a pseudo-commutative square to $\mathsf{Ar}_\triangle\mathcal{M}$ if and only if c sends

a pseudo-commutative square to $\mathsf{Ar}_\triangle\mathcal{E}$. The squares resulting from this are **distinguished squares** depicted as

$$\begin{tikzcd} A \arrow[r, tail] \arrow[d, two heads] \arrow[dr, phantom, "\square"] & B \arrow[d, two heads] \\ C \arrow[r, tail] & D. \end{tikzcd}$$

(K) Corresponding to every $\mathcal{M}$-morphism there is a distinguished square of the form

$$\begin{tikzcd} \varnothing \arrow[r, tail] \arrow[d, two heads] \arrow[dr, phantom, "\square"] & B/A \arrow[d, two heads, "c(f)"] \\ A \arrow[r, tail, "f"] & B \end{tikzcd}$$

and to every $\mathcal{E}$-morphism, a distinguished square of the form

$$\begin{tikzcd} \varnothing \arrow[r, tail] \arrow[d, two heads] \arrow[dr, phantom, "\square"] & A \arrow[d, two heads, "g"] \\ B\backslash A \arrow[r, tail, "k(g)"] & B \end{tikzcd}$$

where $\varnothing$ denotes the (shared) initial object.

Remark 13.7. To augment intuition, one may think of B/A as a cokernel and $B\backslash A$ as a kernel. To relate this definition back to that of the CGW categories introduced in Talk 12, consider the double subcategory of $\mathcal{C}$ whose morphisms are distinguished squares. This can be shown to form a CGW category. One may also check that any regular CGW category satisfying the axioms (D) and (K) in the g-CGW category condition is indeed a g-CGW category with good squares given by the morphisms of $\mathsf{Ar}_\triangle\mathcal{M}$ and $\mathsf{Ar}_\triangle\mathcal{E}$, where all squares are distinguished.

Remark 13.8. The examples introduced previously for CGW categories are also examples of g-CGW categories. There is also a large collection of examples formed by to-be defined **finitely extensive** categories. The structure of such a category is well exhibited by the example of finite sets, so we will first present that example in order to gain some intuition before defining finitely extensive categories in general.

Example 13.9. Let FinSet denote the category of finite sets. It may be viewed as a double category with $\mathcal{M} = \mathcal{E} = \{\text{injective functions of sets}\}$. To see the structure of a g-CGW category on FinSet, let both the good and the pseudo-commutative squares be pullback squares, and the initial object be the empty set. Since $\mathcal{E} = \mathcal{M}$, one may define k and c to take an injective map $A \to B$ to the map $B\backslash A$. Distinguished squares are determined by this data, and thus trace out pushout squares corresponding to objects of $\mathcal{E}$ and $\mathcal{M}$.

Definition 13.10 (Finitely extensive categories)**.** A category $\mathcal{A}$ is **finitely extensive** if it has finite coproducts obeying the following:

- Monic coproduct inclusions.
- Any cospan of the form

$$A \hookrightarrow A \sqcup B \hookleftarrow B$$

has a pullback given by $\varnothing$:

$$\begin{tikzcd} \varnothing \arrow[r] \arrow[d] & A \arrow[d, hook] \\ B \arrow[r, hook] & A \sqcup B. \end{tikzcd}$$

- For any morphism $X \to A \sqcup B$, there exist pullback squares of the form

$$\begin{tikzcd} Y \arrow[r, hook] \arrow[d] & X \arrow[d] & Z \arrow[l, hook'] \arrow[d] \\ A \arrow[r, hook] & A \sqcup B & B \arrow[l, hook'] \end{tikzcd}$$

where $X \cong Y \sqcup Z$.

Remark 13.11. Notice that many of the common categories that have cropped up as examples are finitely extensive categories, such as finite sets, small categories, and topological spaces. In the mould of the example above of FinSet, it turns out that any finitely extensive category can be viewed as a g-CGW category.

Example 13.12. Let $\mathcal{A}$ be a finitely extensive category. The g-CGW structure on $\mathcal{A}$ is defined as follows:

- $\mathcal{M} = \mathcal{E} = \{$coproduct inclusions$\}$.
- Pseudo-commutative and good squares are pullback squares.
- k and c take a coproduct inclusion to the opposite coproduct inclusion, called the **complementary** coproduct inclusion – i.e. in the cospan

$$A \hookrightarrow A \sqcup B \hookleftarrow B$$

both k and c would take

$$A \hookrightarrow A \sqcup B \quad \text{to} \quad A \sqcup B \hookleftarrow B \ .$$

- The distinguished squares are of the form

$$\begin{tikzcd} A \arrow[r, tail] \arrow[d, two heads] & A \sqcup B \arrow[d, two heads] \\ C \sqcup A \arrow[r, tail] & C \sqcup A \sqcup B, \end{tikzcd}$$

which is validated by checking the g-CGW axioms.

One may show that (Z), (M), (G), (D), and (K) hold here.

Example 13.13. Another is the category $\mathsf{Var}_{/k}$ of algebraic varieties, which forms a g-CGW category with

- $\mathcal{M} = \{\text{closed immersions}\}$, $\mathcal{E} = \{\text{open immersions}\}$.
- Pseudo-commutative and good squares are pullback squares.[22]
- Distinguished squares are pullback squares of the form

$$\begin{tikzcd} A \arrow[r, tail] \arrow[d, two heads] \arrow[dr, phantom, "\square"] & B \arrow[d, two heads, "f"] \\ C \arrow[r, tail, "g"] & D \end{tikzcd}$$

  such that $\operatorname{im} f \cup \operatorname{im} g = D$.

One may again check that the axioms of a g-CGW category hold for this setup.

Definition 13.14 ($\star$-CGW categories)**.** A **$\star$-CGW category** is a g-CGW category fulfilling as well the axioms

(GS) The $\mathcal{M}$-square

$$\begin{tikzcd} \cdot \arrow[r, tail, "f"] \arrow[d, tail, "g"] & \cdot \arrow[d, tail, "h"] \\ \cdot \arrow[r, tail, "k"] & \cdot \end{tikzcd}$$

is a good square from f to k if and only if it is such from g to h. The analogous axiom holds for $\mathcal{E}$-squares.

($\star$) Let

$$C \longleftarrowtail A \rightarrowtail B$$

be a diagram in $\mathcal{M}$. Consider the category of good squares, with objects good squares like

$$\begin{tikzcd} A \arrow[r, tail] \arrow[d] \arrow[dr, phantom, "g"] & B \arrow[d, tail] \\ C \arrow[r, tail] & D \end{tikzcd}$$

and morphisms maps $D \rightarrowtail D'$ that commute over B and C. If this category is non-empty, then it has an initial object $D = B \star_A C$, and the induced maps

$$B/A \rightarrowtail B \star_A C/C$$

and

$$C/A \rightarrowtail B \star_A C/B$$

[22]Note that varieties are closed under pullbacks.

are isomorphisms. This statement for the case of the first of these two isomorphisms is depicted as

$$\begin{tikzcd}
A \arrow[r, tail] \arrow[d, tail] \arrow[dr, phantom, "g"] & B \arrow[d, tail] & B/A \arrow[l, two heads] \arrow[d, tail, "\cong"] \\
C \arrow[r, tail] & B \star_A C & B \star_A C/C. \arrow[l, two heads]
\end{tikzcd}$$

The analogous relationships hold for $\mathcal{E}$-diagrams.

(PO) The category of $\mathcal{M}$ good squares corresponding to a diagram

$$C \leftarrowtail A \rightarrowtail B \ ]$$

is nonempty.

Crucially, this *does not have to hold* for $\mathcal{E}$-diagrams.

(PBL) Pullback lemma: if the the exterior square in the following diagram is a pseudo-commutative square, then so is the left square.

$$\begin{tikzcd}
A \arrow[r, tail] \arrow[d, two heads] & B \arrow[r, tail] \arrow[d, two heads] & C \arrow[d, two heads] \\
A' \arrow[r, tail] & B' \arrow[r, tail] & C.
\end{tikzcd}$$

The analogous statement holds for $\mathcal{E}$-morphisms and -compositions.

(POL) Pushout lemma: if the exterior square in the following diagram is a good square, then so is the right square.

$$\begin{tikzcd}
A \arrow[r, tail] \arrow[d, tail] & B \arrow[r, tail] \arrow[d, tail] & D \arrow[d, tail] \\
C \arrow[r, tail] & B \star_A C \arrow[r] & E.
\end{tikzcd}$$

The analogous statement holds for $\mathcal{E}$-morphisms and compositions in the cases where the $\star$-pushout (from ($\star$)) exists.

The asymmetry with respect to $\mathcal{E}$-morphisms and $\mathcal{M}$-morphisms in this definition is key. To foreshadow, it gives a more restricted category of which one can describe the K-theory of common examples.

Example 13.15. Extensive categories, already seen to be g-CGW categories, can also be considered as $\star$-CGW categories where for each span

$$C \sqcup A \leftarrowtail A \rightarrowtail A \sqcup B$$

the corresponding $\star$-pushout is the triple coproduct $C \sqcup B \sqcup A$.

Remark 13.16. This previous example means that many common examples such as finite sets and topological spaces can also be framed in terms of the structure of $\star$-CGW categories. The asymmetry in the definition of $\star$-CGW categories is encapsulated by the following example.

Example 13.17. Consider $\mathsf{Var}_{/k}$, the category of varieties over k. This is a g-CGW category, and can be interpreted as a $\star$-CGW category by setting $\star$-pushouts as pushouts of varieties. Again with $\mathcal{M}$-morphisms as closed immersions and $\mathcal{E}$-morphisms as open immersions, one notices that while axiom (PO) holds for closed immersions, it does not for the $\mathcal{E}$-morphisms since pushouts of open immersions of varieties do not generally exist.

13.2 K-theory

As indicated in Talk 12, one may look at both Quillen's Q-construction and Waldhausen's $S_\bullet$-construction of K-theory for CGW categories. One of the advantages of the latter is that it tracks homotopical data via weak equivalences. It would thus be nice to have a $S_\bullet$-construction recognizing the structure of $\star$-CGW categories, as well. To implement this, we require some notion of weak-equivalence in the context of $\star$-CGW categories.

Definition 13.18 (Acyclic structure on a $\star$-CGW category)**.** Let $\mathcal{C}$ be a $\star$-CGW category. An **acyclicity structure** on $\mathcal{C}$ is a class of objects of $\mathcal{C}$ known as **acyclic objects** such that the following axioms are satisfied.

(IA) Any initial object in $\mathcal{C}$ is an acyclic object.

(A23) Let
$$A \rightarrowtail B \twoheadrightarrow C$$
be a kernel-cokernel pair. A, B, and C satisfy a two-out-of-three property with respect to acyclicity: if any two of them are acyclic objects, then so is the third.

Remark 13.19. Recall from homological algebra the notion of acyclicity: for $F : \mathcal{P} \to \mathcal{Q}$ a functor, the ***F*-acyclic objects** are those $p \in \mathrm{Ob}(\mathcal{P})$ such that the right derived functors of F on them vanish. Acyclic objects can be shown to form a full subcategory of the $\star$-CGW category $\mathcal{C}$ and some morphisms of such objects will play the role of weak equivalences there.

Definition 13.20 (FCGWA categories)**.** Let $\mathcal{C}$ be a $\star$-CGW category and $\mathcal{W}$ its full subcategory of weak equivalences. The pair $(\mathcal{C}, \mathcal{W})$ is called an **FCGWA category**: a functorial CGW category with acyclics.

Definition 13.21 (ECGW functors)**.** Let $(\mathcal{C}, \mathcal{W})$ and $(\mathcal{C}', \mathcal{W}')$ be two FCGWA categories. An **ECGW functor**, $F : (\mathcal{C}, \mathcal{W}) \to (\mathcal{C}', \mathcal{W}')$, is a functor of $\star$-CGW categories that preserves acyclic objects.

Definition 13.22 (Weak equivalences in ECGW categories)**.** Let $\mathcal{C}$ be an ECGW category. An $\mathcal{M}$-morphism is a **weak equivalence** if its cokernel is acyclic and an $\mathcal{E}$-morphism is a **weak equivalence** if its kernel is acyclic.

Remark 13.23. Proceeding we will only cite some major results relating to K-theory that hold in the context of $\star$-CGW categories, and suggest reading the primary reference for details.

There is a notion of **delooping**, as seen in the following theorem:

Theorem 13.24 (Delooping)**.** *The* K*-theory of* $(\mathcal{C}, \mathcal{W})$ *a CGW category forms a spectrum* $\mathsf{K}(\mathcal{C}, \mathcal{W})$*.*

We also have a notion of **fibrations**:

Theorem 13.25 (Fibration)**.** *Let* $\mathcal{C}$ *be a* $\star$*-CGW category with acyclicity structures* $\mathcal{V}$ *and* $\mathcal{W}$ *such that* $\mathcal{V} \subseteq \mathcal{W}$*. Then the following is a homotopy fiber sequence:*

$$\mathsf{K}(\mathcal{W}, \mathcal{V}) \longrightarrow \mathsf{K}(\mathcal{C}, \mathcal{V}) \longrightarrow \mathsf{K}(\mathcal{C}, \mathcal{W}).$$

Finally, we also have a notion of **localization**:

Theorem 13.26 (Localization)**.** *Let* $\mathcal{A}$ *and* $\mathcal{B}$ *be* $\star$*-CGW categories such that* $\mathcal{A} \subseteq \mathcal{B}$ *is a full inclusion of* $\star$*-CGW categories and* $\mathcal{A}$ *is closed under kernels of* $\mathcal{E}$*-morphisms, cokernels of* $\mathcal{M}$*-morphisms, and extensions in* $\mathcal{B}$*. Then there exists* $(\mathcal{B}, \mathcal{A})$ *an ECGW category such that the following is a homotopy fiber sequence:*

$$\mathsf{K}(\mathcal{A}) \longrightarrow \mathsf{K}(\mathcal{B}) \longrightarrow \mathsf{K}(\mathcal{B}, \mathcal{A}).$$

Remark 13.27. Note that this follows directly from the fibration theorem, where $\mathcal{W}$ is replaced by $\mathcal{A}$, $\mathcal{C}$ is replaced by $\mathcal{B}$, and $\mathcal{V}$ is replaced by $\varnothing$.

13.3 Chain complexes of finite sets

Remark 13.28. A significant portion of [SS21] focuses on a construction of the K-theory of (finite) chain complexes over an extensive category. It turns out that the category of chain complexes over any extensive category possesses the structure of a $\star$-CGW category, and a subcategory of such chain complexes, the **exact chain complexes**, can be viewed as the acyclic objects, so consequently there is an $S_\bullet$-construction for the K-theory thereof. Moreover, there is a version of the Gillet-Waldhausen theorem for chain complexes over an extensive category, which gives a way of describing the K-theory of the underlying category in terms of that of the chain complexes over it.

Definition 13.29 (Chain complexes over extensive categories)**.** Let χ be an extensive category. A **chain complex over** χ is a diagram

$$\cdots X_{i+1} \twoheadleftarrow \bar{X}_{i+1} \longrightarrow X_i \twoheadleftarrow \bar{X}_i \longrightarrow X_{i-1} \cdots$$

over χ and for all $i \in \mathbb{N}$, such that the **chain condition** is satisfied, i.e. the following is a pseudo-commuting square:

$$\begin{tikzcd}
\varnothing \arrow[r] \arrow[d, two heads] & \bar{X}_i \arrow[d, two heads] \\
\bar{X}_{i+1} \arrow[r] & X_i
\end{tikzcd}$$

Some terminology:

- The $\{X_i\}$ objects are the **degrees** of X.
- The $\{\bar{X}_i\}$ are the **images** of the X_i.
- Each span
$$X_i \twoheadleftarrow \bar{X}_i \longrightarrow X_{i-1}$$
is a **differential** of X.

These chain complexes form a category denoted $\mathsf{Ch}(\chi)$.

Definition 13.30 (Bounded complexes)**.** A **bounded chain complex** over χ is a chain complex as above such that only finitely-many objects and images are non-empty.

The category of bounded chain complexes will be denoted by $\mathsf{Ch}^b(\chi)$.

Theorem 13.31. *Let χ be an extensive category. The category of chain complexes over χ, $\mathsf{Ch}(\chi)$ form a $\star$-CGW category.*

Remark 13.32. The pieces of this structure for $\mathsf{Ch}(\chi)$ are as follows:

- $\mathcal{M}$-morphisms in $\mathsf{Ch}(\chi)$, **chain $\mathcal{M}$-morphisms** are a collection $\{f_i, \bar{f}_i\}$ of $\mathcal{M}$-morphisms in χ fulfilling
$$\begin{tikzcd}
X_i \arrow[d, tail, "f_i"] & \bar{X}_i \arrow[l, two heads] \arrow[r] \arrow[d, tail, "\bar{f}_i"] & X_{i-1} \arrow[d, tail, "f_{i-1}"] \\
Y_i & \bar{Y}_i \arrow[l, two heads] \arrow[r] & Y_{i-1}
\end{tikzcd}$$
such that the square in $\mathcal{M}$ commutes.
- **Chain $\mathcal{E}$-morphisms** are a collection of $\mathcal{E}$-morphisms $\{g_i, \bar{g}_i\}$ in χ fulfilling
$$\begin{tikzcd}
X_i \arrow[d, two heads, "g_i"] & \bar{X}_i \arrow[l, two heads] \arrow[r] \arrow[d, two heads, "\bar{g}_i"] & X_{i-1} \arrow[d, two heads, "g_{i-1}"] \\
Y_i & \bar{Y}_i \arrow[l, two heads] \arrow[r] & Y_{i-1}
\end{tikzcd}$$
such that the square in $\mathcal{E}$ commutes.

- Pseudo-commutative squares are degree-wise pseudo-commutative squares in χ commuting with all squares of the adjacent morphisms in the same subcategory (i.e. in $\mathcal{E}$ or in $\mathcal{M}$).
- Good squares of $\mathcal{E}$- or $\mathcal{M}$-morphisms are level-wise commuting good squares of chain $\mathcal{E}$- or $\mathcal{M}$-morphisms.

Note that all of this holds for $\mathsf{Ch}^b(\chi)$, as well.

Recall that acyclic objects were instrumental in defining weak equivalences in a $\star$-CGW category. The $S_\bullet$-construction of K-theory is able to handle this more nuanced data, and so to implement this, it would be nice to have a class of acyclic objects in $\mathsf{Ch}(\chi)$. These will come from the following:

Definition 13.33 (Exact chain complexes in extensive categories)**.** Let χ be an extensive category. An **exact chain complex** over χ is a chain complex of the form

$$X_{i+1} \twoheadleftarrow \bar{X}_{i+1} \rightarrowtail X_i \twoheadleftarrow \bar{X}_i \rightarrowtail X_{i-1}$$

such that for each i each mixed cospan

$$\bar{X}_{i+1} \twoheadrightarrow X_i \leftarrowtail \bar{X}_i$$

is a kernel-cokernel pair, and the chain condition is given by the following pseudo-commutative square.

$$\begin{tikzcd}
\varnothing \arrow[r, tail] \arrow[d, two heads] & \bar{X}_i \arrow[d, two heads] \\
\bar{X}_{i+1} \arrow[r, tail] & X_i.
\end{tikzcd}$$

The collection of exact chain complexes is denoted by $\mathsf{Ch}_E(\chi)$.

Theorem 13.34. *$\mathsf{Ch}_E(\chi)$ is a full double subcategory of $\mathsf{Ch}(\chi)$.*

Proposition 13.1. *$(\mathsf{Ch}(\chi), \mathsf{Ch}_E(\chi))$ forms an ECGW category, where $\mathsf{Ch}_E(\chi)$ is the full subcategory of acyclic objects.*

Remark 13.35. This ECGW category restricted to bounded chain complexes actually provides a model for the K-theory of the underlying extensive category χ, a relationship described by a Gillet-Waldhausen theorem in this setting.

Theorem 13.36. *There is a homotopy equivalence between the K-theory of χ with isomorphisms and that of the ECGW category $(\mathsf{Ch}^b(\chi), \mathsf{Ch}^b_E(\chi))$, which has quasi-isomorphisms induced by the acyclic objects, i.e. those in $\mathsf{Ch}^b_E(\chi)$:*

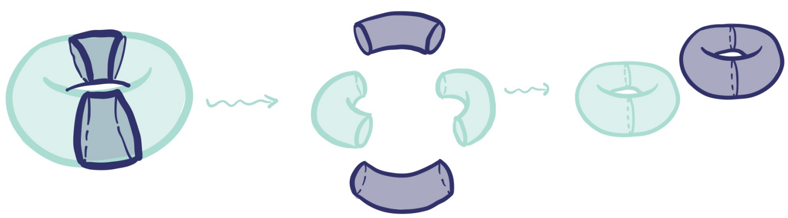
$$\mathsf{K}(\chi) \simeq \mathsf{K}(\mathsf{Ch}^b(\chi), \mathsf{Ch}^b_E(\chi)).$$

14 Talk 14: Square K-theory and manifold invariants (Maxine Elena Calle)

Reference: [Hoe+22].

14.1 Cut-and-paste manifolds

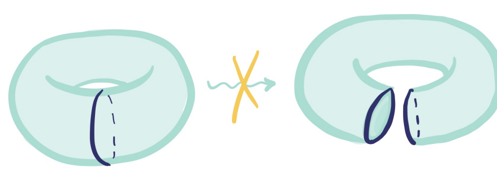
We've talked about scissors congruence for polytopes, and we can try to do the same thing for other kinds of spaces, like manifolds. We can (carefully) cut our manifold up into pieces, rearrange them, and paste them back together. This is called a SK-move (SK comes from *schneiden und kleben*, which means cut and paste in German), and two manifolds are scissors congruent or SK-equivalent if one can be obtained from the other by a finite sequence of these SK-moves.



Example: $T^2 \sim_{SK} T^2 \coprod T^2$

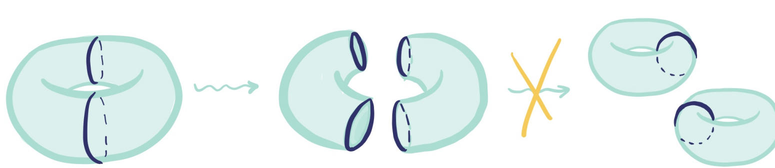
What does it mean to "carefully cut" a manifold M? Here are some non-examples/things to be aware of:

- Our cut must separate M into two (not necessarily connected) pieces M_0 and M_1.



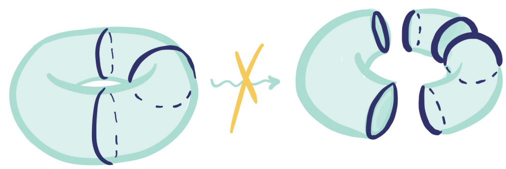
Non-example: cut doesn't separate

- When pasting them back together, we need to glue the boundary of M_0 to the boundary of M_1.



Non-example: glued ∂M_i to itself.

- Among other things, this means that we must have $\partial M_0 \cong \partial M_1$.



Non-example: no way to write $\partial M_0 \cong \partial M_1 \cong (S^1)^{\amalg 3}$.

Example 14.1. In the first picture for $T^2 \sim_{SK} T^2 \coprod T^2$, we cut along four circles to separate T^2 into two (disconnected pieces): M_0 consists of one dark cylinder and one light cylinder, and M_1 consists of the two remaining cylinders. The boundaries of M_0 and M_1 are both diffeomorphic to $(S^1)^{\amalg 4}$, and we glue the dark piece of M_0 to the dark piece of M_1, and similarly for the light pieces. This satisfies all the rules for an SK-move for manifolds.

Here's the formal definition:

Definition 14.2 (SK-moves)**.** An SK**-move** on (smooth, closed, oriented) manifolds is defined as follows: cut an n-manifold M along a codimension-1 smooth submanifold N with trivial normal bundle which separates[23] M. Then paste the two pieces back together along an orientation-preserving diffeomorphism of N.

Definition 14.3 (SK-move)**.** The SK**-group** for n-manifolds is

$$\mathbb{Z}[\text{diffeomorphism classes of } n\text{-manifolds}]/\sim_{SK}$$

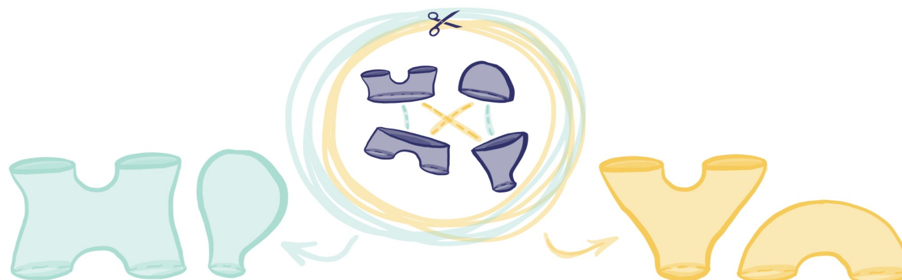
Remark 14.4. The group SK_n can also be defined by a universal property. Let $\mathcal{M}_n$ denote the monoid of diffeomorphism classes of (smooth, closed, oriented) n-manifolds under disjoint union. Then SK_n is defined to satisfy the property that any Abelian group-valued map out of $\mathcal{M}_n$ which is a SK-invariant (i.e. respects SK-equivalence) must factor through it. The only SK-invariants for smooth oriented manifolds are the Euler characteristic, the signature, and their linear combinations.

Fact 14.5. It's a group (under disjoint union).

Remark 14.6. Because we are working with diffeomorphism classes, we don't have access to things like length, angles, scaling (which we needed for the Dehn invariant). This shows us already that scissors congruence of manifolds has a very different flavor than scissors congruence of polytopes!

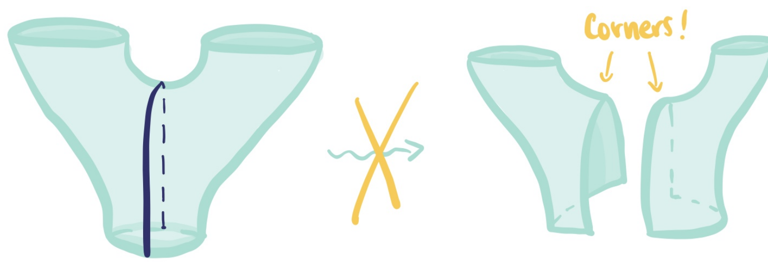
[23]This means $M \backslash N$ is a disjoint union of two components, each with boundary diffeomorphic to N.

Here's another difference from the world of polytopes: when we cut up a polytope, the pieces are all still polytopes, but when we cut up a manifold, we leave the category of manifolds and enter into the category of manifolds *with boundary*. This motivates us to work entirely in the setting of manifolds with boundary. The SK-relation is defined for manifolds with boundary just as it was for manifolds without boundary, with the additional condition that the codimension-1 separating manifold N must not intersect the boundary of M.

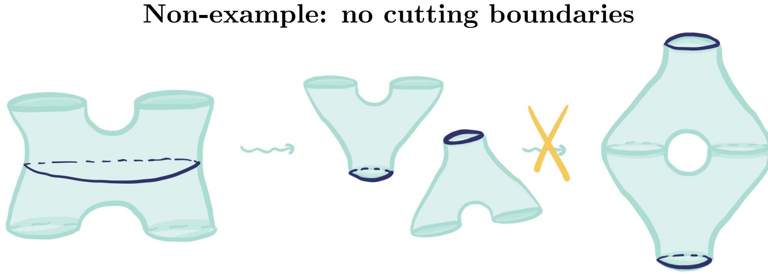


Example: two SK-equivalent manifolds with boundary

Remark 14.7. It is crucial that boundaries are *not allowed* to be cut in these SK-moves, and all boundaries which come from a cut must be pasted back together.



Non-example: no cutting boundaries



Non-example: cuts must be reglued

There are other variants of the definition of SK-moves in the literature which allow for these sorts of things (see [Hoe+22, Remark 2.6]).

Definition 14.8 (SK_n^∂)**.** The group SK_n^∂ is the Grothendieck group (group completion) on the monoid of diffeomorphism classes of (smooth, compact, oriented) n-manifolds with boundary under disjoint union, modulo the SK-relation.

Explicitly,

$$SK_n^\partial = \mathbb{Z}[\text{diffeomorphism classes } [M] \text{ of } n\text{-manifolds with boundary}]/\sim$$

where the relations are generated by

(i) $[M \coprod N] = [M] + [N]$,

(ii) Given manifolds M, M' with closed submanifolds $\Sigma \subseteq M$, $\Sigma' \subseteq M'$ and orientation-preserving diffeomorphisms $\phi, \psi \colon \Sigma \to \Sigma'$,

$$[M \cup_\phi \overline{M'}] = [M \cup_\psi \overline{M'}],$$

where $\overline{M'}$ is M' with the opposite orientation.

This group looks like it should be K_0 of something...and this is exactly what R. Hoekzema, M. Merling, L. Murray, C. Rovi, and J. Semikina show in [Hoe+22] using the of machinery *square* K*-theory*.

Theorem 14.9.

$$SK_n^\partial \cong \mathsf{K}_0^\square(\mathsf{Mfld}_n^\partial).$$

Remark 14.10. Our goal for this talk is to understand this theorem. The benefit of using K-theory is we have access to more structure; the higher K-groups of Mfld_n^∂ can be interpreted as "higher scissors congruence groups" for manifolds with boundary. As of now, there is no known way to realize SK_n as K_0 of some category.

Remark 14.11. In [Hoe+22, Theorem 2.1], the authors show there is a short exact sequence

$$0 \to SK_n \xrightarrow{i} SK_n^\partial \xrightarrow{\partial} C_{n-1} \to 0.$$

Here, C_{n-1} is the group of diffeomorphism classes of nullbordant[24] $(n-1)$-manifolds, i is the inclusion $M \mapsto (M, \emptyset)$, and ∂ sends $(N, \partial N)$ to its boundary ∂N. In fact, this short exact sequence splits and so $SK_n^\partial \cong SK_n \oplus C_{n-1}$. In the "$SK$-book" [Kar+73], they compute

$$SK_n \cong \begin{cases} 0 & n \text{ odd}, \\ \mathbb{Z}[S^n] & n \equiv 2 \pmod 4, \\ \mathbb{Z}[S^n] \oplus \mathbb{Z}[\mathbb{CP}^{n/2}] & n \equiv 0 \pmod 4. \end{cases}$$

Exercise 14.12. Compute $SK_2 \cong \mathbb{Z}[S^2]$ using the classification of surfaces. Hint: we have already seen that $[T^2] = 0$. Show that $[T^2 \# T^2] + [S^2] = 2[T^2] = 0$, so $[T^2 \# T^2] = -[S^2]$, and proceed by induction.

14.2 Square K-theory

Remark 14.13. We have discussed how higher algebraic K-theory can be constructed in settings where we have some way to "chop things up":

exact categories	**Waldhausen categories**	**CGW categories**
short exact sequences	cofiber sequences	spans⇝squares
$0 \to A \to B \to C \to 0$	$A \rightarrowtail B \twoheadrightarrow B/A$	$\begin{tikzcd} A \arrow[r, tail] \arrow[d, dashed, two heads] \arrow[dr, phantom, "\square"] & B \arrow[d, two heads] \\ \bullet \arrow[r, dashed, tail] & C \end{tikzcd}$
Q	$S_\bullet$	Q
$[B] = [A] + [C]$	$[B] = [A] + [B/A]$	$[B] = [A] + [C] - [\bullet]$

We talked about how we can interpret the K-theory of CGW categories as the "combinatorial" analogue of the algebraic K-theory of exact categories. In this talk, we'll develop the combinatorial analogue of the algebraic K-theory of Waldhausen categories: square K-theory.

Categories with squares
squares
$\begin{tikzcd} A \arrow[r, tail] \arrow[d, two heads] \arrow[dr, phantom, "\square"] & B \arrow[d, two heads] \\ C \arrow[r, tail] & D \end{tikzcd}$
$C^{(\bullet)}$
$[D] = [B] + [C] - [A]$

As the name suggests, our setting will be *categories with squares* and we will decompose objects according to these squares. The square K-theory space is built using something similar to the $S_\bullet$-construction, denoted $C^{(\bullet)}$. This is forthcoming work of Campbell-Zakharevich.

We've already seen the idea that "$[D] = [B] + [C] - [A]$" from the square

$$\begin{tikzcd} A \cap B \arrow[r] \arrow[d] & B \arrow[d] \\ A \arrow[r] & A \cup B \end{tikzcd} .$$

An exercise in a first set theory class may be to prove

$$|A \cup B| = |A| + |B| - |A \cap B| .$$

In general, the squares under consideration may not be pushouts or pullbacks (or both, like the example above), but it is a helpful intuition to keep in mind.

[24] By nullbordant M, we mean that $M = \partial W$ for some W.

The benefit of a category with squares is that we get to specify exactly what kinds of squares we want to work with, subject to a few conditions.

Definition 14.14 (Category with squares)**.** A **category with squares** consists of a category $\mathcal{C}$ with coproducts, a chosen distinguished object 0, and two subcategories $c\mathcal{C}$ and $f\mathcal{C}$ called cofibrations ($\rightarrowtail$) and cofiber ($\twoheadrightarrow$) maps along with a collection of distinguished squares

$$\begin{tikzcd} A \arrow[r, tail] \arrow[d, two heads] \arrow[dr, phantom, "\square"] & B \arrow[d, two heads] \\ C \arrow[r, tail] & D \end{tikzcd}$$

which satisfy the following:

(i) distinguished squares are closed under coproducts:

$$\text{if} \quad \begin{tikzcd} A \arrow[r, tail] \arrow[d, two heads] \arrow[dr, phantom, "\square"] & B \arrow[d, two heads] \\ C \arrow[r, tail] & D \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} A' \arrow[r, tail] \arrow[d, two heads] \arrow[dr, phantom, "\square"] & B' \arrow[d, two heads] \\ C' \arrow[r, tail] & D' \end{tikzcd}, \quad \text{then} \quad \begin{tikzcd} A \coprod A' \arrow[r, tail] \arrow[d, two heads] \arrow[dr, phantom, "\square"] & B \coprod B' \arrow[d, two heads] \\ C \coprod C' \arrow[r, tail] & D \coprod D' \end{tikzcd},$$

(ii) distinguished squares are commutative in $\mathcal{C}$ and can be composed vertically and horizontally,

(iii) the subcategory $iso\mathcal{C}$ of isomorphisms ($\xrightarrow{\sim}$) is contained in both $c\mathcal{C}$ and $f\mathcal{C}$,

(iv) all squares of the form

$$\begin{tikzcd} A \arrow[r, tail] \arrow[d, two heads, "\sim"'] & B \arrow[d, two heads, "\sim"] \\ C \arrow[r, tail] & D \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} A \arrow[r, tail, "\sim"] \arrow[d, two heads] & B \arrow[d, two heads] \\ C \arrow[r, tail, "\sim"'] & D \end{tikzcd}$$

are distinguished.

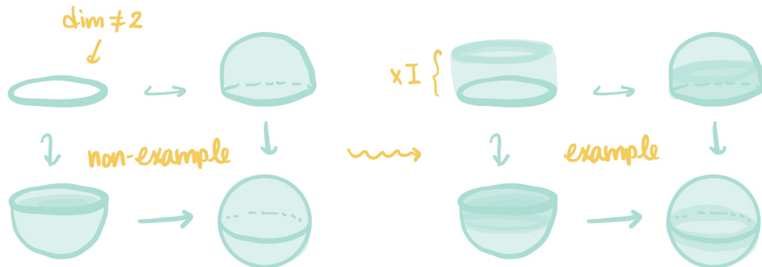
Example 14.15. The objects of Mfld_n^∂ are (nice) n-manifolds with boundary and the morphisms are closed embeddings plus a condition[25] on the boundary. Both $c\mathsf{Mfld}_n^\partial$ and $f\mathsf{Mfld}_n^\partial$ are all morphisms. Distinguished squares are pushouts

$$\begin{tikzcd} N \arrow[r, tail] \arrow[d, two heads] \arrow[dr, phantom, "\square"] & M \arrow[d, two heads] \\ M' \arrow[r, tail] & M' \cup_N M \end{tikzcd}.$$

The distinguished object for Mfld_n^∂ is the empty manifold $\emptyset$.

[25]Specifically a map $M \to M'$ must map each component of ∂M either entirely into the interior of M' or diffeomorphically onto a component of $\partial M'$.

Remark 14.16. Note that the requirement that $M' \cup_N M$ be a *smooth manifold* imposes restrictions on our squares, some of which may feel unfamiliar.



(Non-)example: "thicken" S^1 to get distinguished pushout square

Definition 14.17 ($\mathcal{C}^\bullet$)**.** Define a simplicial category $\mathcal{C}^\bullet$ where $\mathcal{C}^{(n)}$ is the subcategory of $\mathbf{Cat}([n], \mathcal{C})$ whose objects are length n cofibration sequences

$$c_0 \rightarrowtail c_1 \rightarrowtail \cdots \rightarrowtail c_n$$

and whose morphisms are natural transformations in which every square is distinguished. Then $N_*\mathcal{C}^\bullet$ is a bisimplicial set whose (m, n)-simplices look like diagrams

$$\begin{tikzcd}
c_{00} \arrow[r, tail] & \dots \arrow[r, tail] & c_{0m} \arrow[d, two heads] \\
\vdots \arrow[u, leftarrow, two heads] \arrow[d, two heads] & \ddots & \vdots \arrow[d, two heads] \\
c_{n0} \arrow[r, tail] & \dots \arrow[r, tail] & c_{nm}
\end{tikzcd}$$

of distinguished squares.

Recall the *number 1 fact about bisimplicial sets*:

$$|[m] \mapsto |N_m\mathcal{C}^\bullet|| \cong |diag(N_*\mathcal{C}^\bullet)| \cong \left|[n] \mapsto \left|N_*\mathcal{C}^{(n)}\right|\right|.$$

That is, it does not matter whether we realize horizonally then vertically, or vice versa, since both are homeomorphic to the realization of the diagonal simplicial set $[n] \mapsto N_n\mathcal{C}^{(n)}$.

Definition 14.18 (The K-theory space)**.** The **K-theory space** is this realization (with a shift)

$$\mathsf{K}^\square(\mathcal{C}) = \Omega\, |N_*\mathcal{C}^\bullet|\,,$$

and its K-groups are the homotopy groups of $\mathsf{K}^\square(\mathcal{C})$

$$\mathsf{K}_i^\square(\mathcal{C}) = \pi_i(\mathsf{K}^\square(\mathcal{C})).$$

Example 14.19. Every Waldhausen category $\mathcal{C}$ is naturally a category with squares. In fact, there are sometimes multiple ways to do this:

(1) When $w\mathcal{C} = \mathrm{iso}\,\mathcal{C}$, then we can take $c\mathcal{C} = co\mathcal{C}$, $f\mathcal{C}$ =cofiber maps, $\square$ =all commutative squares, $0 = 0$.

(2) No matter what $w\mathcal{C}$ is, we can take $c\mathcal{C} = co\mathcal{C}$, $f\mathcal{C}$ =all maps, $\square$ =pushouts up to weak equivalence,[26] $0 = 0$.

In both cases, a comparison at the level of simplicial objects shows $\mathsf{K}^{\square}(\mathcal{C}) \simeq \mathsf{K}^{W}(\mathcal{C})$.

Typically, K_0 of a category can be described very concretely the free Abelian group on objects modulo some relations (often as the Grothendieck group of some monoid). Campbell-Zakharevich prove that this works for square K-theory as long as we put some (reasonable) assumptions on $\mathcal{C}$.

Theorem 14.20. *Suppose $\mathcal{C}$ is a category with squares, with distinguished object O. If*

- *O is initial or terminal in $c\mathcal{C}$,*
- *O is initial or terminal in $f\mathcal{C}$,*
- *for all objects $A, B \in \mathcal{C}$, there is an object $X \in \mathcal{C}$ so that the squares*

$$\begin{tikzcd} O \arrow[r, tail] \arrow[d, two heads] \arrow[dr, phantom, "\square"] & B \arrow[d, two heads] \\ A \arrow[r, tail] & X \end{tikzcd} \quad \textit{and} \quad \begin{tikzcd} O \arrow[r, tail] \arrow[d, two heads] \arrow[dr, phantom, "\square"] & A \arrow[d, two heads] \\ B \arrow[r, tail] & X \end{tikzcd}$$

  are distinguished,

then

$$\mathsf{K}_0^{\square}(\mathcal{C}) \cong \mathbb{Z}[\mathrm{Ob}\mathcal{C}]/\sim$$

where $\sim$ is generated by $[O] = 0$ and $[A]+[D] = [B]+[C]$ for every distinguished square

$$\begin{tikzcd} A \arrow[r, tail] \arrow[d, two heads] \arrow[dr, phantom, "\square"] & B \arrow[d, two heads] \\ C \arrow[r, tail] & D \end{tikzcd}.$$

Their proof (forthcoming) is very similar to the proof for the Q-construction, basically showing that $\mathsf{K}_0^{\square}(\mathcal{C})$ has the right generators and relations. The assumptions in this theorem are pretty reasonable to ask for (note the similarities with CGW categories), and hence make $\mathsf{K}_0^{\square}$ computable in many cases.

Exercise 14.21. For a Waldhausen category $\mathcal{C}$, recall that $\mathsf{K}_0^{W}(\mathcal{C})$ is the free Abelian group on objects modulo the relation $[A] = [A']$ for every weak equivalence $A \xrightarrow{\sim} A'$ and $[B] = [A]+[B/A]$ for every cofiber sequence $A \rightarrowtail B \twoheadrightarrow B/A$. Using Example 14.19(2), show that $\mathsf{K}_0^{\square}(\mathcal{C}) \cong \mathsf{K}_0^{W}(\mathcal{C})$ directly.

[26] This means that a square is distinguished when $B \cup_A C \xrightarrow{\sim} D$ is a weak equivalence.

Hint: consider the square

$$\begin{tikzcd} A \arrow[r, tail] \arrow[d, two heads] \arrow[dr, phantom, "\square"] & B \arrow[d, two heads] \\ 0 \arrow[r, tail] & B/A \end{tikzcd}$$

and the two cofiber sequences

$$A \rightarrowtail B \twoheadrightarrow B/A$$
$$C \rightarrowtail D \twoheadrightarrow D/C.$$

14.3 Main theorems

Theorem 14.22.
$$\mathsf{K}_0^\square(\mathsf{Mfld}_n^\partial) \cong SK_n^\partial.$$

Proof idea. Since the distinguished object $\emptyset$ is initial in both $c\mathsf{Mfld}_n^\partial$ and $f\mathsf{Mfld}_n^\partial$, we may apply Theorem 14.20.[27] This gives a description of $\mathsf{K}_0^\square(\mathsf{Mfld}_n^\partial)$ in terms of specific generators and relations and we can show that SK_n^∂ is described by the same generators and relations. $\square$

The authors of [Hoe+22] also show that the Euler characteristic lifts (as an SK-invariant) to the level of K-theory.

Theorem 14.23. *There is a map of square* K*-theory*

$$\mathsf{K}^\square(\mathsf{Mfld}_n^\partial) \to \mathsf{K}(\mathbb{Z})$$

which on π_0 *agrees with the Euler characteristic* $\chi\colon \mathrm{SK}_\partial^n \to \mathbb{Z}$.

Proof idea. To prove the theorem, the authors use the intermediary category $\mathsf{Ch}_\mathbb{Z}^{\mathrm{hb}}$ consisting of homologically bounded chain complexes.[28] Recall that $\mathsf{Ch}_\mathbb{Z}^{\mathrm{hb}}$ has the structure of a Waldhausen category, where cofibrations are level-wise injective maps and weak equivalences are quasi-isomorphisms. By Example 14.19(2), we can also give $\mathsf{Ch}_\mathbb{Z}^{\mathrm{hb}}$ the structure of a category with squares. The map $S\colon \mathsf{Mfld}_n^\partial \to \mathsf{Ch}_\mathbb{Z}^{\mathrm{hb}}$ is just the *singular chain functor* which sends a compact manifold with boundary to its singular chain complex. There are two things to show:

(1) S is a map of categories with squares,

(2) $\mathsf{K}(\mathsf{Ch}_\mathbb{Z}^{\mathrm{hb}}) \simeq \mathsf{K}(\mathbb{Z})$ in such a way that S corresponds to χ on π_0.

For (1), the trickiest part is showing that a diffeomorphism $C \cup_A B \xrightarrow{\sim} D$ implies $S(C) \cup_{S(A)} S(B) \xrightarrow{\sim} S(D)$ is a quasi-isomorphism. The idea is to model

[27] Noting that the third condition is clearly satisfied by disjoint union.

[28] I.e. quasi-isomorphic to bounded complexes of finitely-generated $\mathbb{Z}$-modules.

the pushout using $S(B+C)$[29] and use things from Hatcher to show the inclusion $S(B+C) \to S(D)$ is a quasi-isomorphism. One thing to note here is that the choice of distinguished squares in $\mathsf{Ch}_{\mathbb{Z}}^{\mathrm{hb}}$ is crucial for the proof, which would not have worked if we only allowed cofiber maps as the vertical maps.

For (2), the authors use various theorems of higher algebraic K-theory to show that all the maps

$$\mathsf{Mod}_{\mathrm{fg}}^{\mathrm{proj}}(\mathbb{Z}) \xrightarrow{i} \mathsf{Mod}_{\mathrm{fg}}(\mathbb{Z}) \xrightarrow{t} \mathsf{Ch}_{\mathbb{Z}}^{b} \xrightarrow{j} \mathsf{Ch}_{\mathbb{Z}}^{\mathrm{hb}}$$

realize to isomorphisms on K-theory. Here, $\mathsf{Mod}_{\mathrm{fg}}^{\mathrm{proj}}(\mathbb{Z}) \xrightarrow{i} \mathsf{Mod}_{\mathrm{fg}}$ is the inclusion of projective finitely generated $\mathbb{Z}$-modules into finitely generated ones, t maps a finitely generated $\mathbb{Z}$-module A to the bounded chain complex with A in degree 0 and 0's everywhere else, and j is the inclusion of bounded complexes into homologically bounded ones. The final step is to show that the inverse of this map coincides with the Euler characteristic on K_0. □

[29]For each n, $S_n(B+C)$ is the subgroup of $S_n(D)$ consisting on n-chains in D which are sums of n-chains in B and n-chains in C.

15 Talk 15: Cathelineau and Milnor K-theory (Juan Diego Rojas)

References: [Cat03; Cat04; Gon96].

15.1 The Goncharov conjecture revisited

It's time to remind everyone about Goncharov's conjectures from Talk 0.

Notation. Let $V^n = \mathbb{E}^n, \mathbb{S}^n, \mathbb{H}^n$ be one of our favorite geometries and let $\mathcal{P}(V^n)$ be the scissors congruence group.

Remark 15.1. Recall that we defined the (generalized) Dehn invariant

$$\begin{aligned} D_V^n \colon \mathcal{P}(V^n) &\to \oplus_{i=1}^{n-2} \mathcal{P}(V^i) \otimes \mathcal{P}(\mathbb{S}^{n-i-1}) \\ [P] &\mapsto \sum_{i=1}^{n-2} \sum_{\substack{i\text{-dimensional} \\ \text{faces } A}} [A] \otimes [S(A)] \end{aligned}$$

where $S(A)$ is the spherical polytope constructed as follows: take A and then intersect $A^\perp$ with a sphere to get $S(A^\perp)$, then $S(A) = P \cap S(A^\perp)$.

Example 15.2. The map $\mathcal{P}(\mathbb{E}^3) \to \mathcal{P}(\mathbb{E}^1) \otimes \mathcal{P}(\mathbb{S}^1)$ is the original Dehn invariant defined in Talk 0

$$P \mapsto \sum_{\text{edges } e} l(e) \otimes \theta(e).$$

Theorem 15.3 (Goncharov)**.**

$$\begin{tikzcd} \Big(\ker(D_H^{2n-1})|_{\mathcal{P}(H^{2n-1};\overline{\mathbb{Q}})}\Big)_{\mathbb{Q}} \arrow[rr] \arrow[dr, "\mathrm{vol}"'] & & (\mathsf{K}_{2n-1}(\overline{\mathbb{Q}})_{\mathbb{Q}} \otimes \varepsilon(n))^- \arrow[dl, "r^{\mathrm{Bor}}"] \\ & \mathbb{R} & \end{tikzcd}$$

Upshot 15.4. Borel's theorem, which is a theorem for number fields and a conjecture for $\mathbb{C}$, asserts that the **Borel regulator** r^{Bor} is injective up to torsion. Hence if the top horizontal map in the diagram is also injective, then so is the volume map. In other words:

Slogan 15.5. Volume and Dehn invariant separate scissors congruence classes.

That is, the volume map is injective when restricted to the kernel of the Dehn invariant, so if two polytopes have the same volume and Dehn invariant then they are necessarily scissors congruent.

Conjecture 15.6. There is a commutative diagram

$$\begin{tikzcd}
\left(\ker(D_H^{2n-1})\right)_{\mathbb{Q}} \arrow[rr] \arrow[dr, "\mathrm{vol}"'] & & (\mathrm{gr}_n^{\gamma}\mathsf{K}_{2n-1}(\mathbb{C})_{\mathbb{Q}} \otimes \varepsilon(n))^{-} \arrow[dl, "r^{\mathrm{Bor}}"] \\
& \mathbb{R} &
\end{tikzcd}$$

It is worth noting that $\mathrm{gr}_n^{\gamma}\mathsf{K}_{2n-1}(\mathbb{C})$ and $\mathsf{K}_{2n-1}(\bar{\mathbb{Q}})$ are isomorphic up to torsion given Suslin's rigidity conjecture. This explains why Goncharov's theorem actually motivates this conjecture.

Definition 15.7 (The Dehn complex)**.** The **Dehn complex** is

$$\begin{aligned}
D_V^*(n)\colon \mathcal{P}(V^{2n-1}) &\to \bigoplus_{n_1+n_2=n} \mathcal{P}(V^{n_1}) \otimes \mathcal{P}(\mathbb{S}^{n_2-1}) \to \dots \\
\dots \to & \bigoplus_{n_1+\dots+n_k=n} \mathcal{P}(V^{n_1}) \otimes \mathcal{P}(\mathbb{S}^{n_2-1}) \otimes \dots \otimes \mathcal{P}(\mathbb{S}^{n_k-1}).
\end{aligned}$$

The Dehn invariant makes $\bigoplus_n \mathcal{P}(\mathbb{S}^{2n-1})$ into a coalgebra and $\bigoplus_n \mathcal{P}(V^{2n-1})$ is a comodule over this coalgebra. This construction amounts to taking a cobar complex with coefficients in this comodule.

Remark 15.8. In the previous conjecture, we see the first homology appearing, which inspires us to extend the conjecture to other homological degrees:

Conjecture 15.9. There is a map

$$H^i(D^*(n))_{\mathbb{Q}} \to (\mathrm{gr}_n^{\gamma}\mathsf{K}_{2n-i}(\mathbb{C})_{\mathbb{Q}} \otimes \varepsilon(n))^{-}.$$

Moreover, Goncharov conjectures that not only does this map exists, but that it is in fact an isomorphism.

Remark 15.10. Conjectures in motivic cohomology imply existence, but showing it is an isomorphism is much harder. Why did Goncharov think this conjecture is true? He knew that the $H^n(D^*(n))$ case was true, and we now know that the $H^{n-1}(D^*(n))$ case is also true – the rest is wishful thinking. Today we will look at the two known cases.

15.2 Cohomology of the Dehn complex

We will need a bunch of gadgets, and for convenience we will just work in the spherical setting. Recall the previous setup:

- $E = \mathbb{R}^n$,
- $\mathcal{T}(n)$ is the Tits building of $\mathbb{R}^n$,
- $H_{n-2}(\mathcal{T}(n), \mathbb{Q}) = \mathrm{St}(n)$,[30]

[30] We will always work over $\mathbb{Q}$.

- $\mathcal{P}(\mathbb{S}^{n-1}) = H_0(\mathrm{O}_n, \mathrm{St}(n)^t)$.

Construction 15.1. Consider the map $[P] \mapsto L_p(n)$ where $L_p(n)$ is the $\mathbb{Q}$-vector space generated by nonzero $E_1, \ldots, E_p \subseteq E$ with $E = E_1 \oplus \cdots \oplus E_p$. This forms a semisimplicial set whose i^{th} face map is given by

$$(E_1, \ldots, E_p) \mapsto (E_1, \ldots, E_i \oplus E_{i+1}, \ldots, E_p).$$

The realization of this semi-simplicial set is the **Tits building**, suspended so that its homology lives in degree n (where we want it), and so that $H_n(L_*(n)) = \mathrm{St}(n)$.

Remark 15.11. Note that this forms a *semisimplicial* set and not a simplicial set – there are no degeneracy maps because we require the subspaces E_i to be nonzero, so we can't just insert zeros. One could obtain a *simplicial* set by allowing the E_i to be zero.

Definition 15.12 (Orthogonal algebra)**.** Let $A = \bigoplus_{n,i} H_i(\mathrm{O}_n, \mathbb{Q}^t)$ where $\mathbb{Q}^t$ has action twisted by the determinant. This forms an algebra under multiplication in the following way: first map

$$H_i(\mathrm{O}_n, \mathbb{Q}^t) \otimes H_j(\mathrm{O}_m, \mathbb{Q}^t) \to \bigoplus_{n_1+n_2=n} H_{i+j}(\mathrm{O}_n \times \mathrm{O}_m, \mathbb{Q}^t \otimes \mathbb{Q}^t)$$

using Eilenberg-Zilber and Künneth, and then compose with $\mathrm{O}_n \times \mathrm{O}_m \to \mathrm{O}_{n+m}$ to land in $H_{i+j}(\mathrm{O}_{n+m}, \mathbb{Q}^t)$.

Remark 15.13. We are also interested in the homology of Steinberg modules as a coalgebra,

$$\mathbb{H}\mathrm{St} = \bigoplus_{n,q} H_q(\mathrm{O}_n, \mathrm{St}(n)^t).$$

To view this as a coalgebra, use Construction 15.1: Define

$$L_p(n) \to \bigoplus_{\substack{n_1+n_2=n \\ p_1+p_2=p}} L_{p_1}(n_1) \otimes L_{p_2}(n_2)$$

by $(E_1, \ldots, E_p) \mapsto \sum (E_1, \ldots, E_{p_1}) \otimes (E_{p_1+1}, \ldots, E_p)$. This induces a map of complexes $L_*(n) \to \bigoplus_{n_1+n_2} L_*(n_1) \otimes L_*(n_2)$. Now take homology of both sides:

$$\mathrm{St}(n) \cong H_n(L_*(n)) \to \bigoplus_{n_1+n_2} H_n(L_*(n_1) \otimes L_*(n_2)) \cong \bigoplus_{n_1+n_2=n} \mathrm{St}(n_1) \otimes \mathrm{St}(n_2).$$

The last isomorphism follows by Künneth and the fact that the Tits building only has homology in one degree. Use this map to get

$$\begin{aligned} H_n(\mathrm{O}_n, \mathrm{St}(n)^t) &\to \bigoplus_{n_1+n_2} H_q(\mathrm{O}_n, \bigoplus_{V_1 \oplus V_2 = V} \mathrm{St}(n_1)^t \otimes \mathrm{St}(n_2)^t) \\ &\cong \bigoplus_{n_1+n_2=n} H_q(\mathrm{O}_{n_1} \times \mathrm{O}_{n_2}, \mathrm{St}(n_1)^t \otimes \mathrm{St}(n_2)^t) \end{aligned}$$

by applying Shapiro's Lemma.

Upshot 15.14. $\mathbb{H}\mathrm{St}$ is a coalgebra.[31]

We're interested in the case when $q = 0$. Let

$$\mathbb{H}_0\mathrm{St} \coloneqq \sum_n H_0(\mathrm{O}_n, \mathrm{St}(n)^t) \cong \bigoplus_n \mathcal{P}(\mathbb{S}^{n-1}).$$

The crucial fact is that this is an isomorphism of coalgebras for which comultiplication coincides with the Dehn invariant.[32]

Theorem 15.15 (Cathelineau)**.** *For every n, there is a spectral sequence with*

$$E^2_{-p,q} = H^p_q(\mathbb{H}St, \mathbb{Q})_n \Rightarrow H_{-p+q+n}(\mathrm{O}_n, \mathbb{Q}^t).$$

Remark 15.16. The subscript n in $H^p_q(\mathbb{H}\mathrm{St}, \mathbb{Q})_n$ means take n^{th} degree part. This spectral sequence connects the cohomology of the homology of the Steinberg module (as a coalgebra) to the homology of orthogonal algebra.

The proof begins in worst way possible, by considering a tricomplex:

$$C_\alpha(\mathrm{O}_n, \Omega_{\beta,\gamma}\, L_*(n))^t)$$

where

$$\Omega_{\beta,-\gamma} L_*(n) = \sum_{\substack{n_1+\cdots+n_\gamma=n \\ \beta_1+\cdots+\beta_\gamma=\beta}} H_{\beta_1}(\mathrm{O}_{n_1}, L_{\beta_1}(n_1)^t) \otimes \cdots \otimes H_{\beta_\gamma}(\mathrm{O}_{n_\gamma}, L_{\beta_\gamma}(n_\gamma)^t).$$

The trick is to filter it in two different ways: one to get E_2 page and one to get the E_∞ page.

Corollary 15.16.1. *When $q = 0$, $n = 2m$, and $p = m, m-1$, we have*

$$\begin{aligned} E^2_{0,-m} &\cong H^m(D^*(m)) \cong H_m(\mathrm{O}_{2m}, \mathbb{Q}^t), \\ E^2_{0,-m+1} &\cong H^{m-1}(D^*(m)) \cong H_{m+1}(\mathrm{O}_{2m}, \mathbb{Q}^t). \end{aligned}$$

Remark 15.17. This follows by lower degree computations for homology for orthogonal groups. In order to relate all this back to Goncharov's conjecture, we use Milnor K-theory. To do this, we need to change the base field from $\mathbb{R}$ to $\mathbb{C}$. This muddies the connection to scissors congruence groups, since it's unclear how relate complexified scissors congruence to usual scissors congruence. In this setting, the same theorem is true, but it's unclear exactly what it means.

The idea is to show that the isomorphism is induced by the surjective map

$$\otimes^m H_1(\mathrm{O}_2, \mathbb{Q}^t) \to H_m(\mathrm{O}_m, \mathbb{Q}^t).$$

The first observation is that $\otimes H_1(\mathrm{O}_2, \mathbb{Q}^t) \cong \otimes^m H_1(\mathrm{SO}_2, \mathbb{Q})$. We can study the spectral sequence associated to the extension

$$1 \to \mathrm{SO}_2 \to \mathrm{O}_2 \to \{\pm 1\} \to 1.$$

[31] This talk takes algebra to its limit and we just have to go with it.
[32] This is non-obvious.

The map $\lambda \mapsto \mathrm{diag}(\lambda, \lambda^{-1})$ shows $\mathrm{SO}_2 \cong \mathbb{C}^\times$, so

$$\otimes^m H_1(\mathrm{SO}_2, \mathbb{Q}) \cong \otimes^m H_1(\mathbb{C}^\times, \mathbb{Q}) \cong \otimes^m \mathbb{C}^\times/\mu,$$

where μ denotes roots of unity and we quotient by torsion. This extends to a map of the tensor algebra

$$T(\mathbb{C}^\times, \mu) \to \oplus_m H_m(\mathrm{O}_{2m}, \mathbb{Q}^t).$$

We can show that the kernel of this map is the Steinberg relation, and since Milnor K-theory is $T(\mathbb{C}^\times/\mu)$ modulo the Steinberg relations, the map factors as follows:

$$\begin{tikzcd} T(\mathbb{C}^\times/\mu) \arrow[r, two heads] \arrow[d, dashed] & \otimes_m H_m(\mathrm{O}_{2m}, \mathbb{Q}^t) \\ \mathsf{K}_*^{\mathrm{M}}(\mathbb{C}) \arrow[ur] & \end{tikzcd}$$

Question 15.18. How does this relate to scissors congruence? Unclear.

Remark 15.19. There have been several lies in this talk, because some of the claims in [Gon96] are wrong. What is the state of the theorem he claims, and how much of it is correct? Is it possible to fix or not? For example, one of the maps he considers isn't well-defined, and it would be great if someone could fix it, if possible. As of now, Goncharov's conjectures might better be called "Goncharov's fantasies".

16 Talk 16: Rognes' Rank Filtration and Stable Buildings (Jack Burke)

Reference: [Rog92].

16.1 Combinatorics on $S_\bullet$

Recall the Waldhausen $S_\bullet$-construction: If $\mathcal{C}$ is a category with cofibrations co$\mathcal{C}$ and weak equivalences $w\mathcal{C}$, we can form a simplicial category $S_\bullet\mathcal{C}$ where $S_q\mathcal{C}$ has objects of the form

$$* \rightarrowtail A_0 \rightarrowtail \dots \rightarrowtail A_n \ \in \mathrm{Fun}(\mathsf{Ar}[n], \mathcal{C}).$$

Since $S_\bullet\mathcal{C}$ is itself a Waldhausen category, we can iterate $S_\bullet$-construction. Define the K-theory spectrum $\mathsf{K}(C)_k = \left|wS^k_\bullet\mathcal{C}\right| \in \mathsf{Sp}$.

However, we might be remembering more information than we need to. Define

$$\begin{aligned} r : [q] &\to \mathsf{Ar}[q] \\ j &\mapsto (0 \to j). \end{aligned}$$

Then r^* takes a diagram F on $\mathsf{Ar}[q]$ to a diagram $\sigma r^* F$ on $[q]$:

$$* = \sigma(0) \rightarrowtail \sigma(1) \rightarrowtail \dots \rightarrowtail \sigma(q)$$

Lemma 16.1 (And a slogan)**.** *By the pushout condition in $S_\bullet\mathcal{C}$, we can reduce $S_\bullet\mathcal{C}$ to functors on $[q]$.*

Lemma 16.2. *For each q and $n \geq 1$, we have equivalences of categories*

$$(r^*)^n \colon S^n_q\mathcal{C} \to (r^*S_q)^n\mathcal{C}.$$

Remark 16.3. Similarly, we have equivalence of categories when we restrict to the subcategories of cofibrations and weak equivalences. Note that we are suppressing the multi-index by taking the diagonal of our multisimplicial set. Another important observation is that we have fixed a degree q – we are not saying anything (yet) about a simplicial structure.

Proof idea. The proof is by induction. The base case for $n = 1$ goes as follows: view an object of $S_q\mathcal{C}$ as a diagram $\sigma = r^*F$ in $\mathsf{co}\mathcal{C}$ on $[q]$ such that it looks like the $S_\bullet$-construction:

- $\sigma(0) = *$.
- Choose subquotients $\sigma(k)/\sigma(j)$:

$$\begin{tikzcd} \sigma(j) \arrow[r] \arrow[d] & \sigma(k) \arrow[d] \\ * \arrow[r] \arrow[ur, phantom, "\ulcorner"] & \sigma(k)/\sigma(j) \end{tikzcd}$$

- A cofibration is a commutative diagram:

$$\begin{tikzcd} * \arrow[r, tail] \arrow[d, tail] & \sigma(1) \arrow[r, tail] \arrow[d, tail] & \dots \arrow[r, tail] & \sigma(q) \arrow[d, tail] \\ * \arrow[r, tail] & \tau(1) \arrow[r, tail] & \dots \arrow[r, tail] & \tau(q) \end{tikzcd}$$

  such that each pushout is a cofibration in $\mathsf{co}\mathcal{C}$.

Choosing subquotients gives a canonical quotient cofibration extending the diagram to $\mathsf{Ar}[q]$. We can do something similar for weak equivalences, where all vertical arrows are weak equivalences in $\mathcal{C}$. Any consistent choice of subquotients provides an inverse for $r^*F \to F$, so r^* is an equivalence of categories. $\square$

16.2 Rank filtration on K-theory

Remark 16.4. Let R be a ring with strong invariant dimension.[33] The (Quillen) K-theory of *free* finitely generated R-modules agrees with the K-theory of *projective* finitely generated R-modules in all positive degrees.[34] In degree 0, we just get monomorphism, not an isomorphism. Let $\mathcal{F}(R)$ denote category of finitely generated free R-modules. Then we can define filtration by rank,

$$* \simeq F_0(\mathcal{F}(R)) \subseteq F_1(\mathcal{F}(R)) \subseteq \cdots \subseteq \mathcal{F}(R),$$

and use this to define filtration on K-theory.

Definition 16.5. Define $F_k\mathsf{K}(R)_n \subseteq \mathsf{K}(R)_n$ to be the subcomplex realizing the simplicial full subcategory of $[q] \mapsto wS^n_q\mathcal{F}(R)$ which consists of diagrams which factor through the $F_k\mathcal{F}(R)$. For each n, this gives a filtration on K-theory spaces,

$$* \simeq F_0\mathsf{K}(R)_n \subseteq F_1\mathsf{K}(R)_n \subseteq \cdots \subseteq \mathsf{K}(R)_n.$$

Note that this is diagram in the category of spaces.

Lemma 16.6. *This is in fact a diagram of spectra: the connecting maps*

$$\partial_n : \Sigma\mathsf{K}(R)_n \to \mathsf{K}(R)_{n+1}$$

restrict to connecting maps for the filtration

$$\tilde{\partial}_{n,k} : \Sigma F_k\mathsf{K}(R)_n \to F_k\mathsf{K}(R)_{n+1},$$

and so $\{F_k\mathsf{K}(R)_n\}_{n\geq 0}$ *is a prespectrum.*[35]

[33] In particular, if there is a split injection $R^n \hookrightarrow R^m$, then $n \leq m$, i.e. two R-modules are isomorphic if and only if have same dimension. Most nice rings we would want to think about are like this.

[34] Essentially, the inclusion of free into projective induces covering map on K-theory. For details see Thm A.9.1(C) in Thomason-Trobaugh.

[35] What we mean here by "prespectrum" is a collection of spaces E_n with structure maps $\Sigma E_n \to E_{n+1}$, and when the adjoints of the structure maps are equivalences we call it a spectrum. These are sometimes called spectra and Ω-spectra, respectively, instead.

Proof sketch. The connecting map ∂ takes a suspended q-simplex F to the $(1, q)$-bisimplex
$$(0 \rightarrowtail F) \quad \text{in } wS_\bullet S^n_\bullet \mathcal{F}(R),$$
which preserves rank. □

Definition 16.7. The k^{th} *unstable* K*-theory* of R is $F_k\mathsf{K}(R)$, and the rank filtration of $\mathsf{K}(R)$ is given by

$$* \simeq F_0\mathsf{K}(R) \to F_1\mathsf{K}(R) \to \cdots \to \mathsf{K}(R).$$

Lemma 16.8. *These are useful: the unstable* K*-theories approximate* $\mathsf{K}(R)$*,*

$$\operatorname{colim}_k \pi_i F_k\mathsf{K}(R) \xrightarrow{\cong} \pi_i\mathsf{K}(R).$$

Proof. Omitted, since the proof is straightforward and only a few lines. □

This brings us to our main proposition:

Proposition 16.1. *There is an equivalence of spaces*

$$\frac{F_k\mathsf{K}(R)_n}{F_{k-1}\mathsf{K}(R)_n} \cong \frac{D(R^k)_n}{h\,\mathrm{GL}_k(R)},$$

where the right side in the proof will be constructed in the proof, and there is an equivalence of spectra
$$\frac{F_k\mathsf{K}(R)}{F_{k-1}\mathsf{K}(R)} \simeq \frac{D(R^k)}{h\,\mathrm{GL}_k(R)}.$$

Proof. We construct a simplicial category X'_* whose realization is the space $F_k\mathsf{K}(R)_n/F_{k-1}\mathsf{K}(R)_n$. For $q \geq 0$, the objects of X'_q are diagrams on $(\mathsf{Ar}[q])^n$ in $S^n_q\mathcal{F}(R)$ where the largest module has rank exactly k, together with a base object $*_q$. Morphisms are isomorphisms of such diagrams.

Now let's simplify: let X_q be the category whose objects are lattices[36] on $[q]^n$ on free R-modules with top module isomorphic to R^k, together with a base object $*_q$. By (a filtered version of) the previous lemma, there is an equivalence of categories $r^* : X'_q \xrightarrow{\sim} X_q$. We claim that X_* is a simplicial category and r^* is an equivalence of simplicial categories. The problem is that X_* does not support a $0th$ face map. Recall that d_0 of $S_q\mathcal{C}$ uses the choice of sub-quotients in a diagram on $\mathsf{Ar}[q]$ but r^* forgets these. But we can fix this problem, because we know a non-degenerate σ in X'_q is mapped to $d_0(\sigma) = *_q$, and so we can define d_0 on X_* by taking non-degenerate simplices to the base point $*_q$. This additional structure makes the equivalence of simplicial categories true.

Now we study X_*, which has two relevant simplicial subcategories

$$D(R^k)_n \subseteq Y_* \subseteq X_*,$$

[36] A lattice is the "correct shape" of the diagram in the n-th iterated $S_\bullet$-construction, with pushouts and etc.

defined as follows: Y_k is the full subcategory of X_k whose objects are those lattices where the top module is equal(!) to R^k and the cofibrations are inclusions. The simplicial subcategory $D(R^k)_n$ has the same objects as Y_* but only identity morphisms, and is called a **building**.

Upshot 16.9. Every object in Y_* is isomorphic to an object of X_*, and choosing isomorphisms gives us a deformation retraction $|X| \xrightarrow{\simeq} |Y|$. Moreover, the morphisms in Y_* are determined by a source object and the action of the morphism on R^k, i.e. an element of $\mathrm{GL}_k(R)$, which means Y_* is the simplicial (based) translate category for the $\mathrm{GL}_k(R)$-action on $D(R^k)_n$. Thus we can conclude

$$F_k\mathsf{K}(R)_n/F_{k-1}(R)_n \simeq |X| \simeq |Y| \simeq D(R^k)_n/h\,\mathrm{GL}_k(R).$$

To lift to spectra, we note that the inclusions $D(R^k)_* \subseteq Y_* \subseteq X_*$ respect the structure maps on K-theory.

□

16.3 Barratt-Priddy-Quillen Theorem

Rognes gives a slick proof of this well-known theorem. Let FinSet_* denote the category of finite pointed sets whose cofibrations are injections and weak equivalences are bijections. We can filter FinSet_* by cardinality, which tells us we have a functor $\mathsf{FinSet}_* \to \mathcal{F}(R)$ that respects filtration, given by mapping a finite set I to R^I, the free R-module generated by the set I. We also identify R^I with $R^{|I|}$.

Definition 16.10 (Axial submodules and standard apartments)**.** The **axial submodules** of R^k are those submodules R^I for $I \subseteq \mathbf{k}$ (set with k elements). Let $D^*(\mathbf{k}) \hookrightarrow D^*(R^k)_n$ be the subcomplex of lattice diagrams in axial submodules of R. We call

$$A_{n,k} = D(\mathbf{k})_n \subseteq D(R^k)_n$$

the **standard apartment** inside the building $D(R^k)_n$. One can show there is a homeomorphism $A_{n,k} \cong S^{nk}$.

Corollary 16.10.1 (Barratt-Priddy-Quillen)**.**

$$\mathsf{K}(\mathsf{FinSet}_*) = \mathcal{S},$$

where $\mathcal{S}$ denotes the sphere spectrum.

Proof. Look at the graded piece $F_k\mathsf{K}(\mathsf{FinSet}_*)_n/F_{k-1}\mathsf{K}(\mathsf{FinSet}_*)_n$. Our proposition says this space is equivalent to $D(\mathbf{k})_n/h\Sigma_k \simeq S^{nk}/h\Sigma_1$. When $k=1$, we have $S^n/h\Sigma_1 \cong S^n$. For $k>1$, $S^{nk}/h\Sigma_k$ is $(2n-1)$-connected, which gives us a stable equivalence $\mathcal{S} = F_1\mathsf{K}(\mathsf{FinSet}_*) \hookrightarrow \mathsf{K}(\mathsf{FinSet}_*)$. □

17 Talk 17: Campbell-Zakharevich's Solution to Constructing the Goncharov map (Elise McMahon)

Reference: [CZ19].

17.1 Setup: the Goncharov map

Remark 17.1. Recall from the previous talks that we can iterate the Dehn invariant: for a given X (usually one of our three favorites geometries), the i^{th} Dehn invariant is a map

$$D^i \colon \mathcal{P}(X^n) \to \mathcal{P}(X^i) \otimes \mathcal{P}(\mathbb{S}^{n-i-1}),$$

and so there are two possible ways to apply it again, either to $\mathcal{P}(X^i)$ or to $\mathcal{P}(\mathbb{S}^{n-i-1})$. We use the notation $\mathcal{P}(X) \coloneqq \mathcal{P}(X, I(X))$.

Example 17.2. In dimension 3, there is a commutative diagram:

$$\begin{tikzcd}
\mathcal{P}(X^5) \arrow[r] \arrow[d] & \mathcal{P}(X^3) \otimes \mathcal{P}(\mathbb{S}^1) \arrow[d] \\
\mathcal{P}(X^1) \otimes \mathcal{P}(\mathbb{S}^3) \arrow[r] & \mathcal{P}(X^1) \otimes \mathcal{P}(\mathbb{S}^1)
\end{tikzcd}$$

Remark 17.3. Goncharov constructs the complex $P_*(X)$ from this data. The problem is that homology of homology is hard. Campbell-Zakharevich's solution to this problem is to construct a simplicial set whose homology is $\mathcal{P}(X, 1)$, and then define then Dehn invariant directly on this. The goal of this approach is to delay taking coinvariants $H_0(G, -)$ for as long as possible. This ends up being pretty powerful, since it will allow us to access tools of homotopy theory. Specifically, we'll be able to commute a certain cofiber and homotopy colimit, which will make things easier to analyze.

Recall that Goncharov conjectured the existence of a map

$$H_m(P_*(X^{2n-1})) \to (\mathrm{gr}^\gamma_n \, \mathsf{K}_{n+m}(\mathbb{C})_\mathbb{Q} \otimes \varepsilon(n))^\pm.$$

Campbell-Zakharevich show there is a homomorphism

$$\phi_n \colon H_{n+m}(I(X), \mathbb{Z}[1/2]^\sigma) \to H_m(P_*(X)).$$

Note the differences between this map and the one Goncharov conjectured. In particular, the map goes the other direction! The goal for this talk is to understand this theorem and its proof.

17.2 The RT-building and homology

Remark 17.4. Here's the set up: let X be one of the geometries $\mathbb{S}^n$ or $\mathbb{H}^n$ given by quadratic forms $q = x_0^2 + \cdots + x_n^2$ and $q = -x_0^2 + x_1^2 + \ldots x_n^2$, respectively. A **subspace** U of X is the linear space U' such that $q|_{U'}$ is non-degenerate and has a maximum negative signature. If X is $\mathbb{H}^n$, then U is isometric to some hyperbolic space $\mathbb{H}^i$, then $U^\perp$ is spherical, i.e. isometric to some $\mathbb{S}^{n-i}$. Alternatively if X is $\mathbb{S}^n$ then $U \simeq \mathbb{S}^i$ and $\mathbb{S}^\perp$ is $\mathbb{S}^{n-i}$. Note that the perp spaces are spherical in both cases.

Remark 17.5. We'll need to invert 2 a lot.[37] To realize this geometrically, we'll smash $\mathbb{S}^\sigma$, which denotes $\mathbb{S}^1$ with a twisted action.

Definition 17.6 (RT-buildings)**.** The ***RT*-building**[38] F_*^X is the simplicial set whose i-simplices are chains of non-empty subspaces

$$U_0 \subseteq \cdots \subseteq U_i$$

such that $U_i = X$. The face and degeneracy maps do what we would expect: d_i deletes U_i and s_j repeats U_j.

Fact 17.7. The isometry group $I(X)$ acts on F_*^X by

$$g \cdot (U_0 \subseteq \cdots \subseteq U_i = X) = (gU_0 \subseteq \cdots \subseteq gU_i = gX).$$

This construction does what we want in the sense of the following theorem.

Theorem 17.8. *The map $\mathcal{P}(X,1) \to H_{n+1}(\mathbb{S}^\sigma t \wedge F_*^X)$ given by*

$$\{x_1, \ldots, x_n\} \mapsto \sum_{\sigma \in \Sigma_{n+1}} \operatorname{sgn}(\sigma)[^{(}x_{\sigma(0)} \subseteq^{(} x_{\sigma(0)}, x_{\sigma(1)}) \subseteq \cdots \subseteq^{(} x_{\sigma(0)}, \ldots, x_{\sigma(n)})]$$

is an isomorphism, and so

$$\mathcal{P}(X,G) \cong H_0(G, H_{n-1}(\mathbb{S}^\sigma \wedge F_*^X)).$$

Theorem 17.9. *$H_n(F_*^X) \neq 0$ only in dimension $n = \dim X$.*

Now our goal is to construct the Dehn invariant on F_*^X. The key idea is that (in the spherical and hyperbolic cases) angle can be captured by projecting onto orthogonal complement.

Generalizing this picture, we can use this new idea of "angle" it to help construct the Dehn invariant. To do that, we need to replace tensor products of the polytope algebras with reduced joins of the FT-buildings.

[37] Compare and contrast: Goncharov worked rationally.
[38] Named after Rognes-Tits.

Definition 17.10 (Reduced joins of simplicial sets)**.** For pointed simplicial sets X, Y, the **reduced join** is $X \hat{*} Y$ whose m-simplices are

$$(X \hat{*} Y)_m = \bigvee_{i+j=m-1} X_i \wedge Y_j.$$

The reduced join $X \hat{*} Y$ is weakly equivalent to $\mathbb{S}^\sigma \wedge X \wedge Y$. We'll define the Dehn invariant for single subspace, and then define it more generally by gluing things together.

Definition 17.11 (D_U)**.** Let U be a proper, non-empty subspace of X and define $D_U \colon F_*^X \to F_*^U * F_*^{U^\perp}$ by

$$(U_0 \subseteq \cdots \subseteq U_n) \mapsto \begin{cases} (U_0 \subseteq \cdots \subseteq U_j) \wedge (pr_{U^\perp} U_{j+1} \subseteq \cdots \subseteq pr_{U^\perp} U_n) & j = \max\{i \mid U_i = U\} \\ * & U \neq U_j \text{ for any } j. \end{cases}$$

Definition 17.12 (Derived Dehn invariants)**.** The dimension i **derived Dehn invariant** D_i is the lift

$$\begin{tikzcd} & \bigvee\limits_{\substack{U \subseteq X \\ \dim U = i}} F_*^U * F_*^{U^\perp} \arrow[d] \\ F_*^X \arrow[ur, dashed, "D_i"] \arrow[r] & \prod\limits_{\substack{U \subseteq X \\ \dim U = i}} F_*^U \hat{*} F_*^{U^\perp} \end{tikzcd}$$

Fact 17.13. This map is $I(X)$ equivariant.

Theorem 17.14. *$H_0(I(X), H_{n+1}(\mathbb{S}^\sigma \wedge D_1))$ is the classical Dehn invariant.*

Remark 17.15. Now we want to be able to take all possible iterations of Dehn invariants, which requires knowing that certain diagrams commute.

Lemma 17.16.

$$\begin{tikzcd} F_*^X \arrow[r, "D_i"] \arrow[d, "D_j"'] & \bigvee\limits_{\substack{U \subseteq X \\ \dim U = i}} F_*^U \hat{*} F_*^{U^\perp} \arrow[d, "\hat{*} D_{j-i-1}"] \\ \bigvee F_*^V \hat{*} F_*^{V^\perp} \arrow[r, "D_i \hat{*}"'] & \bigvee F_*^U * F_*^{U^\perp \cap V} * F_*^{V^\perp} \end{tikzcd}$$

Theorem 17.17. *$H_0(I(X), H_{n+1}(\mathbb{S}^\sigma \wedge D))$ is the classical Dehn cube.*

Remark 17.18. Now our goal is to get a spectral sequence[39] which converges to the total homotopy cofiber of something. Then we can analyze this cofiber and make use of the fact that we can swap colimits – specifically we take coinvariants and the total homotopy cofiber $\mathrm{cof}^{\mathrm{th}}$.

[39] The one we saw earlier in previous talks.

Proposition 17.1. *Let $\overline{F}\colon I_n \to \mathsf{Top}_*$ be a functor where I_n is the indexing category for a n-cube. Then there is a spectral sequence with*

$$E^1_{p,q} = \bigoplus_{A=(b,a_1,\ldots,a_{n-p-1})} \tilde{H}_q(\overline{F}(A)) \Rightarrow \tilde{H}_{p+q}(\operatorname{cof}^{ht} \overline{F})$$

The index A is ??? with $|A| = \dim X$ and $b + a_1 + \cdots + a_{n-p-1} = d$.

Example 17.19. In dimension 3, the indexing category is

$$I_3 = \begin{array}{ccc} (3) & \longrightarrow & (2,1) \\ \downarrow & & \downarrow \\ (1,2) & \longrightarrow & (1,1,1) \end{array}.$$

This is keeping track of possible ways we might split up the Dehn invariant starting at dimension 3.

Definition 17.20 (Dehn cube)**.** The **Dehn cube** is the functor $D^0\colon I_n \to \mathsf{Top}_*$ which maps $(b, a_1, \ldots, a_k)$, where $b + a_1 + \cdots + a_k = d$, to

$$\bigvee_{\substack{W\oplus V_1\oplus\cdots\oplus V_k \\ \dim W = b, \dim V_j = a_{j-1}}} F_*^W \hat{*} (\hat{*}_{j=1}^k F_*^{V_j}).$$

Thus

$$E^1_{p,q} = \bigoplus_{A=(b,a_1,\ldots,a_{n-p-1})} \tilde{H}_1(D^0(A)_{hI(X)}, \mathbb{Z}[1/2]).$$

Remark 17.21. This is summing up different ways we can take Dehn invariants, and also $\Rightarrow \tilde{H}_{p+q}((\operatorname{cof}^{\text{th}})_{hI(X)})$.

The bottom non-trivial row ends up being $P * (X)$,

$$\left(\tilde{H}_{d+1}(F_*^X)_{hI(X)}) \to \bigoplus_{|A|=2} \tilde{H}_{d+1}(D^0(A)_{hI(X)})) \to \ldots \right) \Rightarrow P_*(X).$$

The upshot is that we get a projection to the base. Given a spectral sequence $E^*_{p,q} \Rightarrow G_{p+q}$, if $E^1_{*,n}$ is the first non-zero row then we get homomorphisms $\theta_m\colon G_m \to E^1_{m-n,n}$. Applying this fact to our situation yields a homomorphism

$$H_m((\operatorname{cof}^{\text{th}} D)_{hI(X)}, \mathbb{Z}[1/2]) \to H_m(P_*(X)).$$

Now we just need to understand $H_m((\operatorname{cof}^{\text{th}} D)_{hI(X)}, \mathbb{Z}[1/2])$, so we compute $\operatorname{cof}^{\text{th}}(D^0)_{hI(X)}$ in the following extremely important theorem:

Theorem 17.22.

$$\operatorname{cof}^{\text{th}}(D^0)_{hI(X)} \simeq_{[2]} (\mathbb{S}^t \wedge \mathbb{S}^{n-1})_{hI(X)}.$$

With this theorem in hand, we then have

$$\begin{aligned} H_m((\mathrm{cof}^{\mathrm{th}}\, D)_{hI(X)}, \mathbb{Z}[1/2]) &\cong H_m((\mathbb{S}^\sigma \wedge \mathbb{S}^{n-1})_{hI(X)}, \mathbb{Z}[1/2]) \\ &\cong H_{n+m}(I(X), H_n(\mathbb{S}^\sigma \wedge \mathbb{S}^{n-1}; \mathbb{Z}[1/2])) \\ &\cong \tilde{H}_{n+m}(I(X), \mathbb{Z}[1/2]^\sigma) \end{aligned}$$

and using θ_m we get a map

$$\phi_n \colon H_{n+m}(I(X), \mathbb{Z}[1/2]^\sigma) \to H_m(P_*(X)).$$

Remark 17.23. Like in the Goncharov conjecture, we can fit this map into a triangle involving $\mathbb{R}$-volume. The Cheeger-Chern-Simons map plays the role of the Borel regulator.

Conjecture 17.24. This map is an isomorphism.[40]

[40]Inna will bet money that it is not!

18 Talk 18: Some Reflections and Some Questions (Jonathan Campbell)

We began with the classical problem of scissors congruence in $\mathbb{R}^3$, and transitioned to some crazy K-theory and homological ways of defining things. We also discussed combinatorial K-theory, and spent the last day of the workshop on Goncharov's conjectures. Without a global perspective on things, it's difficult to see how all of this fits together, so we will start by discussing the rank filtration and then elaborate on conjectures.

18.1 The rank filtration and review

The rank filtration was originally due to Quillen (we saw the stable rank filtration due to Rognes); this can be found in the paper "On finite generation of $\mathsf{K}_i(O_K)$". The idea is to look at $wS_\bullet\mathsf{Vect}_F^{\mathrm{fin}}$ and filter by dimension

$$wS_\bullet\mathsf{Vect}_F^{\mathrm{fin},\leq n} \to wS_\bullet\mathsf{Vect}_F^{\mathrm{fin}}.$$

The associated graded is given by $\Sigma^2 T(F)$; this is the Tits building!

If we compute the homology of the cofiber sequences involving the graded terms, we obtain a sequence whose cofiber is the homology of $\Sigma^2 T(F)_{h\,\mathrm{GL}_n}$, which gives $H_{i-n}(\mathrm{GL}_n, \mathrm{St}(F))$. As a consequence, the associated graded looks like a GL-version of scissors congruence, and $H_{i-n}(\mathrm{GL}_n, \mathrm{St}(F))$ looks like a Lie group acting discretely on the Steinberg module.

As an example of the applications of this filtration, Lee–et al. computed that $\mathsf{K}_3(\mathbb{Z}) = \mathbb{Z}/48$, along with other calculations about the K-theory of integers. More recently, other authors have determined that $\mathsf{K}_8(\mathbb{Z}) = 0$ as well as some other K-groups of the integers, and they use a very concrete resolution plus computers to do a lot of these calculations.

The objects $H_{i-n}(\mathrm{GL}_n, \mathrm{St}(F))$ keep coming up in the rank filtrations, and so in some sense scissors congruence is equivalent to the study of associated graded to filtrations in K-theory. Can the information flow in both directions? To push this as far as possible, you want to consider different versions of K-theory, and the fact that K-theory is cobbled together from filtrations makes this feasible.

18.2 Complexes and speculation

In this workshop, we've discussed complexes built out of scissors congruence groups. As a reminder, we have a complex

$$\mathcal{P}(\mathbb{E}^3) \xrightarrow{D} \mathcal{P}(\mathbb{S}^1) \otimes \mathcal{P}(\mathbb{E}^1) \to \Omega^1_{\mathbb{R}/\mathbb{Z}}.$$

But what happens if we have complexes involving

- $P_*(\mathbb{E}^{2n-1})$,
- $P_*(\mathbb{S}^{2n-1})$, and

- $P_*(\mathbb{H}^{2n-1})$?

It could be that the relevant complexes would involve the exterior algebra on Kähler differentials, but Jonathan and Inna expect these are wrong. The conjectures surrounding these other complexes should involve the rank filtration as opposed to the weight filtration; the rank filtration gets used in this work by Goncharov.

Additionally, algebraic K-theory is probably not the right thing to work with. Instead, we probably want to work with a variant of Hermitian K-theory because we care about quadratic forms, and Hermitian K-theory records exactly this structure. However, as of now, the existing notions of Hermitian K-theory do not encode the correct structure.[41]

The tail end of $\mathcal{P}(\mathbb{S}^{2n-1})$ has homology given by $\mathsf{K}_*^M(k)_{\mathbb{Q}}$, and the tail end of the complex $\mathcal{P}_*(\mathbb{E}^{2k-3})$ has homology given by $\Omega^{2k-3}_{\mathbb{R}/\mathbb{Z}}$. There is a map between these two complexes by including one face, inducing a map on homology. This turns out to be a map from Milnor K-theory to Kähler differentials (in appropriate degrees), which coincides with dlog, which in turn agrees with the Dennis trace map.

Question 18.1. What is the relationship between these complexes, and can we interpret this in terms of some Dennis trace map? How can we correctly express Goncharov's conjecture in terms of the rank filtration on Hermitian K-theory?

The original motivation for Jonathan's and Inna's work was to resolve the generalized Hilbert's 3rd problem, which says that the generalized Dehn invariant and volume are complete invariants of scissors congruence; i.e. $H_?(\mathcal{P}_*(\mathbb{E}^?)) = \mathbb{R}$. One way to approach resolving the problem is by computing the homology of these complexes. The work by Jonathan and Inna gives you a foothold for this super classical problem.

Upshot: The theme of the week should be that we are cooking up a bunch of fancy tools to solve this really classical concrete problem!

18.3 Weight filtration vs. rank filtration

Recall that we could have defined K-theory using the $+$-construction. This is nice because given the standard K_0-group of representations of GL_n, considered as an algebraic group over $\mathbb{Z}$, we can define maps

$$R_{\mathbb{Z}}\,\mathrm{GL}_n(A) \to [\mathrm{BGL}_n(A)^+, \mathrm{BGL}_n(A)^+]$$

using the universal properties of the $+$-construction. So, passing to the colimit, we get some natural operations on the algebraic K-theory space of A induced from the representation ring of $\mathrm{GL}(A)$. In particular, there are

- lambda operations on K-theory induced by taking the exterior power of a module,

[41] Jonathan suspects there is a rank filtration on a modified Hermitian K-theory, and conjectures Hochschild homology might be involved.

- Adams operations which on line bundles are given by tensor powers,
- Gamma operations (which are mysterious).

The weight filtration is a filtration induced by these gamma operations, but we care about them because the associated graded terms of the gamma filtration, $\mathsf{K}_*^{(i)}(A)$, are eigenspaces for the Adams operations, yielding a decomposition

$$\mathsf{K}_*(A)_{\mathbb{Q}} \simeq \bigoplus \mathsf{K}_*^{(i)}(A)_{\mathbb{Q}}.$$

Remark 18.2. There's a paper by Grayson which talks about Adams operations on the $S_\bullet$-construction, but it is a bit complicated. In general, Grayson's papers are important to read if you are studying K-theory.

There is another version of the unstable rank filtration;

$$\mathrm{Fil}^{\leq r}\,\mathsf{K}_*(A)_{\mathbb{Q}} = \mathrm{im}\,(\mathrm{prim}\,H_*(\mathrm{GL}_r)_{\mathbb{Q}} \to \mathrm{prim}\,H_*(\mathrm{GL})_{\mathbb{Q}})$$

Here are some questions and open problems about this:

- How on earth does this relate to other rank filtrations? How about stable rank filtrations?
- (Weight vs. rank filtration conjecture) Are the associated graded of the weight and rank filtrations are the same?[42]
- How does this relate to the geometric constructions like the polytope algebra?
- What is a homotopical interpretation of the weight filtration? Even if you don't want it for K-theory, what about for some variant of where we have geometry in play, such as Hermitian K-theory?
- Find some easier analogue of the weight and rank filtrations on K-theory, on more geometric versions of K-theory, or the polytope algebra, or ...

18.4 A few open questions about combinatorial K-theory

- Write down the Waldhausen theorems for squares K-theory. [43]
- In K-theory of rings, the Milnor K-theory $\mathsf{K}_*^M(F)$ is the "totally decomposable" part of K-theory in that

  $$F^\times \otimes \cdots \otimes F^\times \to \mathsf{K}_n^M(F) \to \mathsf{K}_n(F),$$

  and you can get your hands on Milnor K-theory more easily. What about if you replace algebraic K-theory with the K-theory of varieties over F?
- Many more open questions we've discussed throughout the week ...

Reminder: We've been focusing on very classical problems (Greek classical, even), and it's been extremely profitable to play around with formal structures, but also to play around with a lot of these classical homotopical problems!

[42] This is known to be true for number fields and that's about it.
[43] If you do, Jonathan will buy you a (very good) beer.

19 Open Questions

Question 19.1 (Liam & Lucy)**.** What can be said about the association $S \mapsto \mathrm{K}(\mathrm{Var}_S)$? In particular

1. Given a field k and a (k,k)-bimodule M, what is the relationship between $\mathrm{K}(\mathrm{Var}_{k\oplus M})$ and $\mathrm{K}(\mathrm{Var}_k)$, where $k \oplus M$ is the trivial square-zero extension.

2. Does the association above satisfy descent with respect to various topologies on schemes, e.g. the Zariski, Nisnevich, étale topologies etc.

3. (Possibly a goofy question) Can we make sense of the K-theory of varieties over a base ring spectrum or simplicial commutative ring?

Question 19.2 (Liam)**.** There is an equivalence $\mathrm{K}(\mathrm{Var}_k) \simeq \mathrm{K}(\mathrm{Sch}_{\mathrm{k}}^{\mathrm{red,ft}})$ by the CGW dévissage theorem. Generalize this result to arbitrary bases S (the expectation is that it should work).

Question 19.3 (Inna)**.** Come up with new proofs that a given rectangle is scissors congruent to another rectangle of the form $1 \times \text{area}$, where area is the area of the rectangle; Inna mentioned this as a challenge in the first talk and that she collects such proofs.

Question 19.4 (Inna)**.** Counterexample or proof of the claim: if we force a Waldhausen category to satisfy the extension axiom, then the associated K-theories are the same.

Question 19.5 (Ming)**.** Let us work over field $\mathbb{C}$. Two observations:

1. Liu-Sebag: Suppose X, Y are two complex varieties such that $[X] = [Y]$ in $K_0(\mathrm{Var})$. Then X is piecewise isomorphic to Y when X has only finitely many rational curves.

2. Slogan from Arithmetic Geometry: the negativity of the canonical line bundle of a complex variety X controls the number of the rational curves in X. This is quantified by Manin's conjecture.

Motivation: we want to understand how combinatorial K-theory may contain geometric information. Can the higher K-groups tell us something about the negativity of the canonical line bundle of complex varieties? Perhaps see if there exists some kind of trace map and an HKR-isomorphism for combinatorial K-theory of varieties? Can we extend K-theoretic techniques used for point-counting to curve-counting? Does K-theory detect e.g. if a complex variety fails to be uniruled?

Question 19.6 (Ming)**.** Observation from Goncharov:

1. All hyperbolic 3-manifolds have Dehn invariant zero.

Problem: determine that all hyperbolic 3-manifolds of same finite volume are scissors congruent, or construct a counter-example.

This is a warm-up to the bigger problem of understanding if hyperbolic volume and the Dehn invariant determine hyperbolic scissors congruence. If there exists hyperbolic 3-manifolds of same volume which are not scissors congruent, why is this? Can we define an algebraic invariant capturing this? How might we extend this invariant to all hyperbolic 3-polyhedra?

Question 19.7 (Liam)**.** Give a comparison theorem between the square K-theory of the category of k-varieties and all the other K-theories we've seen throughout the week; e.g. CGW or assembler K-theories. We know that all the K_0's agree and have the same universal properties imposed by the cut and paste relations.

Question 19.8 (Jonathan)**.** Determine (as many as possible of) the usual Waldhausen K-theory theorems for square K-theory. Jonathan remarked that the square K-theory allows you to do something K-theoretic when you don't have cofibers, since you can pretend they exist by remembering the map. Moreover, this philosophy applies to higher cubical versions of K-theory.

Question 19.9 (Inna)**.** Understand what is precisely true regarding Goncharov's theorems in "Volumes of hyperbolic manifolds and mixed motives" – there is a map in the paper which is not well-defined (I believe it's the volume map), but the status of this work is unknown.

References

[BGN21] Oliver Braunling, Michael Groechenig, and Anubhav Nanavaty. *The Standard Realizations for the K-theory of Varieties*. July 2021. arXiv: 2107.01168 [math.AG].

[Bor15] Lev Borisov. *Class of the Affine Line Is a Zero Divisor in the Grothendieck Ring*. Mar. 2015. arXiv: 1412.6194 [math.AG].

[Cam19] Jonathan A. Campbell. *The K-Theory Spectrum of Varieties*. 2019.

[Cat03] J. Cathelineau. "Scissors Congruences and the Bar and Cobar Constructions". In: *Journal of Pure and Applied Algebra* 181 (2–3 2003), pp. 141–179.

[Cat04] J. Cathelineau. "Projective Configurations, Homology of Orthogonal Groups, and Milnor K-Theory". In: *Duke Math. J.* 121 (2 2004), pp. 343–387.

[CWZ] J. Campbell, J. Wolfson, and I. Zakharevich. *A comparison of constructions of the Grothendieck spectrum of varieties*. In preparation.

[CZ18] Jonathan A. Campbell and Inna Zakharevich. "Devissage and Localization for the Grothendieck Spectrum of Varieties". In: arXiv:1811.08014 (Nov. 2018). arXiv: 1811.08014 [math.KT].

[CZ19] Jonathan Campbell and Inna Zakharevich. "Hilbert's Third Problem and a Conjecture of Goncharov". In: (Oct. 2019). arXiv: 1910.07112 [math.KT].

[DPS88] Johan L Dupont, Walter Parry, and Chih-Han Sah. "Homology of Classical Lie Groups Made Discrete. II. H2, H3, and Relations with Scissors Congruences". In: *Journal of Algebra* 113.1 (Feb. 1988), pp. 215–260. ISSN: 00218693.

[DS82] Johan L. Dupont and Chih-Han Sah. "Scissors Congruences, II". In: *Journal of Pure and Applied Algebra* 25.2 (Aug. 1982), pp. 159–195. ISSN: 00224049.

[Dup01] Johan L. Dupont. *Scissors Congruences, Group Homology, and Characteristic Classes*. Vol. 1. Nankai Tracts in Mathematics. Singapore ; River Edge, NJ: World Scientific, 2001.

[Gon96] Alexander Goncharov. "Volumes of Hyperbolic Manifolds and Mixed Tate Motives". In: (Jan. 1996). arXiv: alg-geom/9601021 [math.AG].

[Goo14] Thomas G. Goodwillie. "Scissors Congruence with Mixed Dimensions". In: (Oct. 2014). arXiv: 1410.7120 [math.KT].

[Hoe+22] Renee S. Hoekzema et al. "Cut and Paste Invariants of Manifolds via Algebraic K-theory". In: *Topology and its Applications* (Apr. 2022), p. 108105. ISSN: 0166-8641.

[Jes68] Børge Jessen. "The Algebra of Polyhedra and the Dehn-Sydler Theorem". In: *Mathematica Scandinavica* 22.2 (1968), pp. 241–256. ISSN: 0025-5521.

[Kar+73] U. Karras et al. *Cutting and pasting of manifolds; SK-groups*. Mathematics Lecture Series 1. Boston, MA: Publish or Perish, Inc., 1973.

[Kar14] Ilya Karzhemanov. *On the cut - and - paste property of algebraic varieties*. 2014. arXiv: 1411.6084 [math.AG].

[LL20] Michael J. Larsen and Valery A. Lunts. "Irrationality of motivic zeta functions". In: *Duke Mathematical Journal* 169.1 (Jan. 2020).

[McM89] Peter McMullen. "The Polytope Algebra". In: *Advances in Mathematics* 78.1 (Nov. 1989), pp. 76–130. ISSN: 0001-8708.

[Nic09] Johannes Nicaise. *An introduction to p-adic and motivic zeta functions and the monodromy conjecture*. 2009. DOI: 10.48550/ARXIV.0901.4225. URL: https://arxiv.org/abs/0901.4225.

[NS11] Johannes Nicaise and Julien Sebag. "The Grothendieck Ring of Varieties". In: *Motivic Integration and Its Interactions with Model Theory and Non-Archimedean Geometry*. Ed. by Johannes Nicaise, Julien Sebag, and Raf Cluckers. Vol. 1. London Mathematical Society Lecture Note Series. Cambridge: Cambridge University Press, 2011, pp. 145–188.

[Qui71] Daniel Quillen. "The Spectrum of an Equivariant Cohomology Ring: I". In: *Annals of Mathematics* 94.3 (1971), pp. 549–572.

[Qui73] Daniel Quillen. "Finite Generation of the Groups K_i of Rings of Algebraic Integers". In: *Higher K-Theories*. Ed. by H. Bass. Lecture Notes in Mathematics. Berlin, Heidelberg: Springer, 1973, pp. 179–198.

[Rog92] John Rognes. "A Spectrum Level Rank Filtration in Algebraic K-theory". In: *Topology* 31.4 (Oct. 1992), pp. 813–845. ISSN: 0040-9383.

[Sah81] Chih-Han Sah. "Scissors Congruences, I: The Gauss-Bonnet Map". In: *Mathematica Scandinavica* 49.2 (1981), pp. 181–210. ISSN: 0025-5521.

[SS21] Maru Sarazola and Brandon Shapiro. "A Gillet-Waldhausen Theorem for Chain Complexes of Sets". In: (July 2021). arXiv: 2107.07701 [math.KT].

[Syd65] J. P. Sydler. "Conditions Nécessaires et Suffisantes Pour l'équivalence Des Polyèdres de l'espace Euclidien à Trois Dimensions". In: *Commentarii Mathematici Helvetici* 40.1 (Dec. 1965), pp. 43–80. ISSN: 1420-8946.

[Voi16] Claire Voisin. "Stable birational invariants and the Lüroth problem". In: *Surveys in Differential Geometry* 21.1 (2016), pp. 313–342.

[Wal85] Friedhelm Waldhausen. "Algebraic K-theory of Spaces". In: *Algebraic and Geometric Topology*. Ed. by Andrew Ranicki, Norman Levitt, and Frank Quinn. Lecture Notes in Mathematics. Berlin, Heidelberg: Springer, 1985, pp. 318–419.

[Wei13] C.A. Weibel. *The K-book: An Introduction to Algebraic K-theory*. Vol. 145. Graduate Studies in Mathematics. American Mathematical Society, 2013.

[Zak] Inna Zakharevich. *On $K_1 of an assembler$*. DOI: 10.48550/ARXIV.1506.06197. URL: https://arxiv.org/abs/1506.06197.

[Zak16] Inna Zakharevich. *The K-Theory of Assemblers*. Sept. 2016. arXiv: 1401.3712 [math].

[Zak17] Inna Zakharevich. "The Annihilator of the Lefschetz Motive". In: *Duke Mathematical Journal* 166.11 (Aug. 2017). ISSN: 0012-7094.