ORDERABILITY AND BRANCHED COVERS

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Outline

1. Left-orderability

2. L-space conjecture

3. Branched Covers

4. A special family

5. Further questions

Def: A group G is left-orderable if there is a strict total order on G with the property that whenever $g<h$ holds then $fg<fh$ also holds for all $f\in G$.

(Non)-EX: No group with torsion can be LO.

Let $g^n=1$ for some $n>1$. Assume WLOG $g>1$.

Then
- $g\cdot g>g\cdot 1 \Rightarrow g^2>1$
- $g^3>g^2$

$\vdots$

- $g^n>1 \Rightarrow 1>1$ $\#$

Remark: We will focus on when $\pi_1(M)$ is LO for M^3. Sometimes we'll write M is LO instead of $\pi_1(M)$.

Conjecture: (Ozsváth-Szabó, Boyer-Gordon-Watson, Juhász)

Suppose M^3 is compact, irreducible and orientable. Then the following are equivalent:

1. M admits a taut foliation
2. M is LO
3. M is not a Heegaard Floer L-space

Heuristic: All of these properties are measuring "complexity" of M.

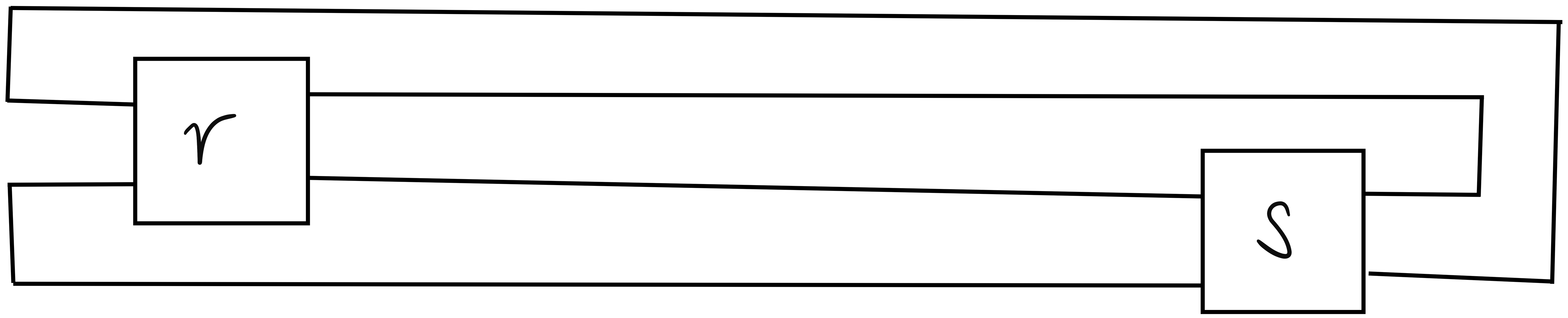
Rmk: There has been a lot of work on M^3 which is surgery on a knot. If M^3 is not surgery on a knot, concluding if M satisfies any of 1-3 is hard.

Def: A branched cyclic cover of M^3 is N^3 so that $M \cong N/\Gamma$ where Γ is cyclic and Γ acts on N by orientation preserving diffeos.

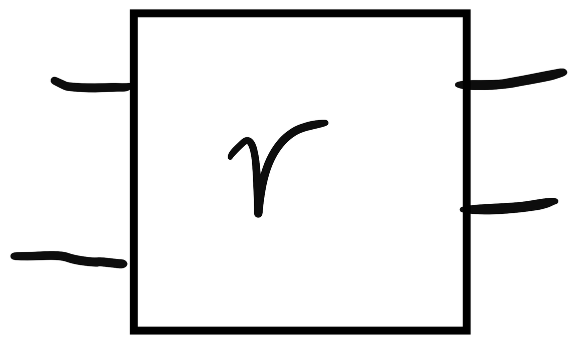
Remark: If this action is free then $N \twoheadrightarrow M$ is a covering space. In this case we'll assume there is $x \in N$ fixed by some $g \in \Gamma$. It can be shown that the fixed set is a 1-mfd.

Def: Let F denote the fixed set of $\Gamma \curvearrowright N^3$ with Γ cyclic, and let L be the image of F in the quotient $N/\Gamma \cong M$. We'll say N is a branched Γ-cover of M over L.

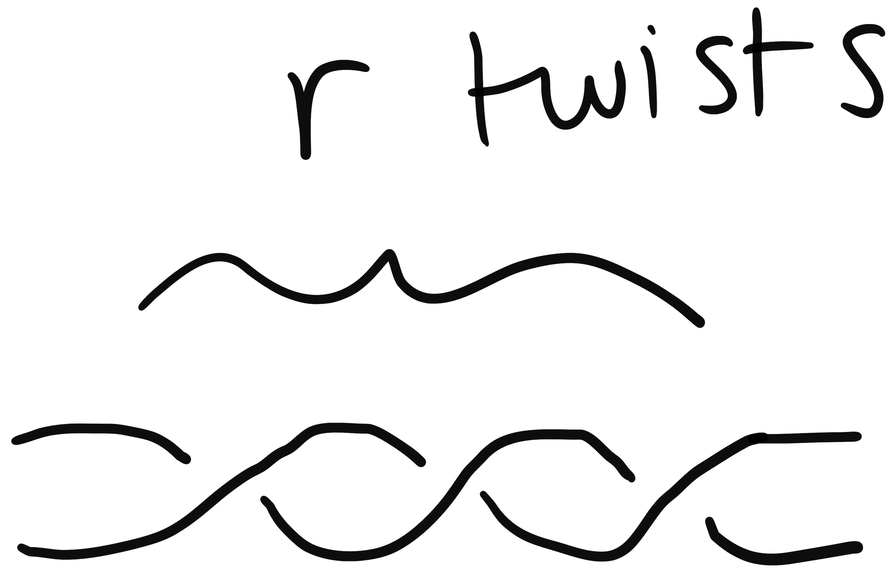
Prop: Let K be a connected 1-submfd (Knot) $\hookrightarrow S^3$. Fix $\Gamma = \mathbb{Z}/n\mathbb{Z}$. For each $n \geq 2 \in \mathbb{Z}$ there is a unique N which is a branched Γ-cover of S^3 over K. We will denote $N = \Sigma_n(K)$ and call it the branched cyclic cover of K of index n.



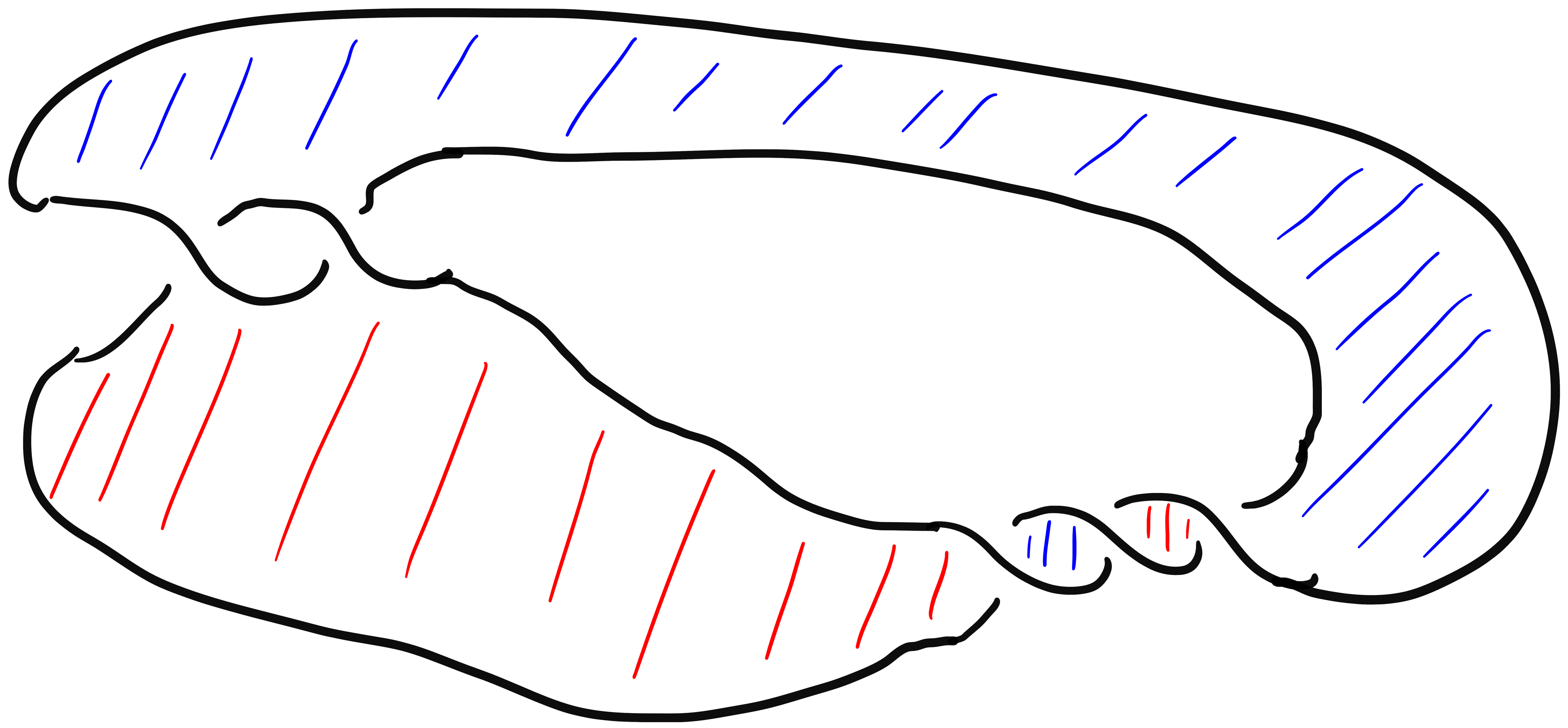
The Knot $J(r,s)$.



means replace the box with



Ex:



The Knot $J(2,3)$

Def The genus of a knot in S^3 is the minimum genus achieved by an orientable surface S with $\partial S = K$. We'll denote this by $g(K)$.

Rmk: ① By combining a theorem of Boyer-Gordon-Wiest and a theorem of Bergman, M^3 with a (nontrivial) $\widetilde{PSL(2,\mathbb{R})}$-rep is LO.

② Work of Hu shows that certain $PSL(2,\mathbb{R})$-reps of $\pi_1(S^3 - J(r,s))$ lift to $\widetilde{PSL(2,\mathbb{R})}$-reps of $\Sigma_n(J(r,s))$.

Thm (Hu) $\Sigma_n(J(r,s))$ [with $g(J(r,s)) > 1$] is LO for $n \gg 0$

Thm (Tran) $\Sigma_n(J(r,s))$ [with $g(J(r,s)) > 1$] is LO for $n \geq f(r,s)$.

Thm (T) $\Sigma_n(J(r,s))$ is LO for

$n \geq 5$ if $g(J(r,s)) = 2$

$n \geq 4$ if $g(J(r,s)) = 3$

$n \geq 3$ if $g(J(r,s)) \geq 4$

Takeaway: Doesn't depend on the parameters so much as the genus.

Further Questions

① What about the other aspects of the L-space conjecture for $\Sigma_n(J(r,s))$?

② What forms can the set

$$\left\{ n \geq 2 \;\middle|\; \begin{array}{l} \Sigma_n(K) \text{ is LO} \\ \Sigma_n(K) \text{ is not an L-space} \\ \Sigma_n(K) \text{ admits a taut foliation} \end{array} \right\}$$

for a fixed Knot $K \hookrightarrow S^3$ take ? To what extent does this depend on the genus?

③ Which M^3 admit $\widetilde{PSL}(2,\mathbb{R})$-reps ?

Thanks for listening!