Stable Homotopy Seminar

The Adams Spectral Sequence

$\mathcal{A}$

Preliminaries

- Work in $\text{hoSp}^{\mathbb{N}}$, aka SHC, with the stable model structure
  - ↳ Fibrant objects are Ω-Spectra, Weak equivalences: π_*-isos.
  - ↳ Write morphisms as $[X,Y]_* := \text{Hom}_{SHC}(X,Y) \in gr^{\mathbb{Z}}Ab$

• p is any prime, we'll fix $p=2$

Agenda
- Construct the AdSS
- Work out the E_2 1-line

- $\mathbb{Z}_{\hat{p}}$: The p-completion of $\mathbb{Z}$ (aka p-adic integers).

  $\mathbb{Z}_{\hat{p}} = \lim\limits_{\leftarrow n} \mathbb{Z}/p^n\mathbb{Z}$

- $\mathbb{Z}_{(p)}$: The p-localization of $\mathbb{Z}$ (aka p-local integers)

  $\mathbb{Z}_{(p)} = S^{-1}\mathbb{Z}, \quad S = p\mathbb{Z}$

  $= \{ \alpha/\beta \in \mathbb{Q} \mid p \nmid \beta \}$

- These are "dual" in the following sense: for $X \in Sp$,

  - $X_{\hat{p}} := L_{M(p)} X, \quad \pi_*(X_{\hat{p}}) = \pi_*(X) \otimes_{\mathbb{Z}} \mathbb{Z}_{\hat{p}}$

  - $X_{(p)} := L_{H\mathbb{F}_p} X \quad \pi_*(X_{(p)}) = \pi_*(X) \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}$

Bousfield Localizations

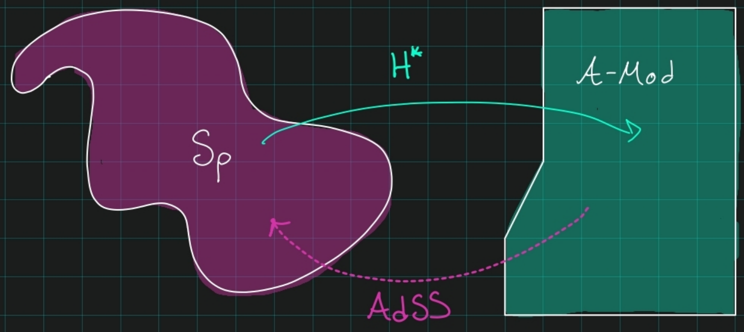
Goals & Motivations

- **Goal:** Compute $[X, Y]$! Take $X = Y = \mathbb{S}$ $\rightsquigarrow$ $[\mathbb{S}, \mathbb{S}] = \pi_*^{st} S^0$ Stable homotopy of spheres
  ↳ Or just $X = \mathbb{S}$ to compute $\pi_* Y$ in Sp.

- **Strategy:**

  - Use $\overbrace{(A \xrightarrow{f} B)}^{Sp} \xrightarrow{H^*} \overbrace{(H^*A \xrightarrow{H^*f} H^*B)}^{gr^{\mathbb{Z}}\mathcal{A}\text{-Mod} \hookrightarrow gr^{\mathbb{Z}}\mathbb{F}_p\text{-Mod}}$

    Use AdSS to recover info



- Construct a spectral sequence converging to $E_\infty^{*,*} \approx [X, Y]_*$
- Identify $E_2 \in \mathbb{F}_p\text{-Mod}$, lifts to $E_2 \in \mathcal{A}\text{-Mod}$ (abelian category!)
  where $\mathcal{A}$ is the mod-p Steenrod algebra

Main focus today

Left exact $\Rightarrow$ right derived, Projectives are F-acyclic for covariant Hom

- Identify E_2 as an $\mathrm{Ext}_{\mathcal{A}}(E_* X, -)$ compute as $\mathbb{R}\mathrm{Hom}_{\mathcal{A}}(E_* X, -)$
- "Resolve" $Y \in Sp$ in a tower $Y_\bullet \xrightarrow{\varepsilon} Y$ where $E_* Y_\bullet \xrightarrow{\varepsilon_*} E_* Y$ is a projective resolution
- Play $\mathcal{A}$, $\mathrm{Ext}_{\mathcal{A}}$, and E_2 structures to find (permanent or not) cycles
- (Hard) Compute differentials & permanent cycles.

Various Adams Spectral Sequences

- Restrict to Sp (f.t., b.b.) $H^n(A) \in \mathbb{Z}\text{-Mod}$ is finitely generated in each degree n and and $\pi_{\leq m} A = 0$ for some $m = m(X)$ (so some finite suspension is connective).

- <u>Most general</u>: for $A, B \in Sp$, E a <u>cohomology</u> theory,

$$E_2^{s,t} = \mathrm{Ext}^{s,t}_{E^*E}(E^*B, E^*A) \Rightarrow [A, L_E B]_{t-s}$$

(for E with Adams condition!)

Convergence depends on E

- Note swapped order of A & B
- Recall $E^*E = E^*(E) = [E, E]_{-*}$

- <u>More specific</u>: take $E = H\mathbb{F}_p$, write $H^*X := H^*_{sing}(X; \mathbb{F}_p)$,

(Cohomological) $$E_2^{s,t} = \mathrm{Ext}^{s,t}_{\mathcal{A}}(H^*B, H^*A) \Rightarrow [A, B_{\hat{p}}]_{t-s}$$

- Using $\mathcal{A} = H\mathbb{F}_p^* H\mathbb{F}_p$ (From Liam's talk)

- <u>Special Case</u>: take $A = B = \mathbb{S}$,

  - Check $H^n\mathbb{S} := [\mathbb{S}, H\mathbb{F}_p]_{-n} = [\Sigma^{-n}\mathbb{S}, H\mathbb{F}_p] = \pi_{-n} H\mathbb{F}_p = \begin{cases} \mathbb{F}_p, & n = 0 \\ 0, & \text{else} \end{cases}$ $\Rightarrow \underline{H^*\mathbb{S} = \mathbb{F}_p}$

  - Check $[\mathbb{S}, L_{H\mathbb{F}_p}\mathbb{S}]_* = [\mathbb{S}, \mathbb{S}_{\hat{p}}]_* = (\pi_*\mathbb{S})_{\hat{p}} = \pi_*\mathbb{S} \otimes_{\mathbb{Z}} \mathbb{Z}_{\hat{p}}$

$$E_2^{s,t} = \mathrm{Ext}^{s,t}_{\mathcal{A}}(\mathbb{F}_p, \mathbb{F}_p) \Rightarrow \pi_*\mathbb{S} \otimes \mathbb{Z}_{\hat{p}}.$$

Constructing the AdSS

• **Def (Adams Tower):** For $Y \in SHC$, an <u>Adams tower</u> for Y is

$$\begin{tikzcd} Y = Y_0 \arrow[d] & Y_1 \arrow[l, "i_0"'] \arrow[d] & Y_2 \arrow[l, "i_1"'] \arrow[d] & Y_3 \arrow[l, "i_2"'] \arrow[d] & \cdots \arrow[l, "i_3"'] \\ J_0 & J_1 & J_2 & J_3 & \end{tikzcd}$$

See Bob's Primer Section 1!

where

✓ 1) $J_n = \mathrm{hocofib}(i_n)$

From Kunneth 2) The induced maps $H^* i_n$ are all zero

3) $H_* J_n \in \mathcal{A}\text{-Mod}(\mathrm{Proj})$

See Amitsur complex (or cobar) 4) There are induced isos $\forall X \in SHC$

$$[X, J_n]_* \xrightarrow[\sim]{H^*} \mathrm{Hom}^*_{\mathcal{A}}(H^* J_n, H^* X)$$
$$[f] \longmapsto H^* f$$

Claim: An Adams tower exists $\forall\, Y \in SHC$.

Constructing the AdSS: Adams Towers

- Consider $\mathrm{hofib}(\varepsilon) \to \mathbb{S} \xrightarrow{\varepsilon} H\mathbb{F}_p$, let $E := H\mathbb{F}_p$, $\bar{E} := \mathrm{cofib}(\varepsilon)$.

- Claim: for $Y = \mathbb{S}$, there is an Adams Tower:

$$\begin{tikzcd}
Y \arrow[d] & \bar{E} \arrow[l, "i_0"'] \arrow[d] & \bar{E}^{\wedge 2} \arrow[l, "i_1"'] \arrow[d] & \bar{E}^{\wedge 3} \arrow[l] \arrow[d] & \cdots \arrow[l] \\
E & E \wedge \bar{E} & E \wedge \bar{E}^{\wedge 2} & E \wedge \bar{E}^{\wedge 3} & \cdots
\end{tikzcd}$$

Triangles bound cofiber sequences

Pf:

$\boxed{\text{Using } \mathbb{S} \wedge A \overset{\sim}{\to} A}$

$$\begin{tikzcd}
\bar{E} \arrow[r, "i_0"] & \mathbb{S} \arrow[r] \arrow[d, "(-\wedge \bar{E})"] & E \arrow[dd, bend left, "(-\wedge \bar{E}^{\wedge n})"] \\
\bar{E}^{\wedge 2} \arrow[r, "i_1"] & \bar{E} \arrow[r] & E \wedge \bar{E} \\
\bar{E}^{\wedge (n+1)} \arrow[r, "i_n"] & \bar{E}^{\wedge n} \arrow[r] & E \wedge \bar{E}^{\wedge n}
\end{tikzcd}$$

Cof. Seq $\in$ SHC

$$\boxed{Y_{n+1} \xrightarrow{i_n} Y_n \xrightarrow{j_n} J_n}$$

$\downarrow [X, -]_*$ Covariant Hom Puppe for Sp

LES $\in gr^{\mathbb{Z}}Ab$

$$[X, Y_{n+1}]_* \xrightarrow{i_n^*} [X, Y_n]_* \xrightarrow{j_n^*} [X, J_n]_* \xrightarrow{\partial_n^*} [X, \Sigma Y_{n+1}]_* \xrightarrow{i_n^*} [X, \Sigma Y_n]_* \xrightarrow{j_n^*} \cdots$$

$\downarrow$ Assemble into an exact couple by summing over n

$$\begin{tikzcd}
\bigoplus_n [X, Y_n]_* \arrow[rr, "\bigoplus_n i_n^*"] & & \bigoplus_n [X, Y_n]_* \arrow[dl, "\bigoplus_n j_n^*"] \\
& \bigoplus_n [X, J_n]_* \arrow[ul, "\bigoplus_n \partial_n^*"] &
\end{tikzcd}$$

Rmk: To get a tower for A, just apply $(- \wedge A)$ everywhere.

Constructing the AdSS: Exact Couples

- How to get a sseq from an exact couple: take <u>derived</u> <u>couple</u>:

$$\overbrace{\begin{tikzcd} A \arrow[rr, "i"] & & A \arrow[dl, "j"] \\ & B \arrow[ul, "k"] & \end{tikzcd}}^{E_0} \quad \overset{H^*(-)}{\rightsquigarrow} \quad \overbrace{\begin{tikzcd} \mathrm{im}(i) \arrow[rr, "\tilde{i}"] & & \mathrm{im}(i) \arrow[dl, "\tilde{j}"] \\ & H^*(B, \partial_1) \arrow[ul, "\tilde{k}"] & \end{tikzcd}}^{(E_1, \partial_1)}$$

$j \circ i,\ k \circ j,\ i \circ k = 0$

Defining Maps

- $\tilde{i} :=$ Restrict i to $\mathrm{im}(i)$
- $\partial : B \to B$, $\partial = j \circ k$
- $\tilde{j} : \mathrm{im}(i) \to H^*B$
  - ↳ For $a' \in \mathrm{im}(i)$, pick $a \in i^{-1}(a')$, set
    Set $\tilde{j}(a') := j(a) \bmod \partial B$
    $= [j(a)]_{H^*B}$
- $\tilde{k} : H^*(B) \longrightarrow \mathrm{im}(i)$
  Set $\tilde{k}(h \bmod \partial B) := k(h)$

↳ Check j' is well-defined: diagram chase on a

Show $a_0, a_1 \in i^{-1}(a') \Rightarrow j(a_0 - a_1) \equiv 0 \bmod \partial B$

↳ Check $\mathrm{im}(\tilde{k}) \subseteq \mathrm{im}(i)$: use exactness of E_0, $\partial b = 0 \Rightarrow (j \circ k)(b) = 0 \Rightarrow b \in \ker(j) = \mathrm{im}(i)$

↳ Check $\partial^2 = 0$: $\partial^2 = j \circ \underbrace{k \circ j}_{=0 \text{ by exactness}} \circ k$

↳ Check exactness of E_2: (long) diagram chase

Set $E_r = \{H^* E_{r-1}, \partial_{r-1}\} \rightsquigarrow$ pages of spectral sequence

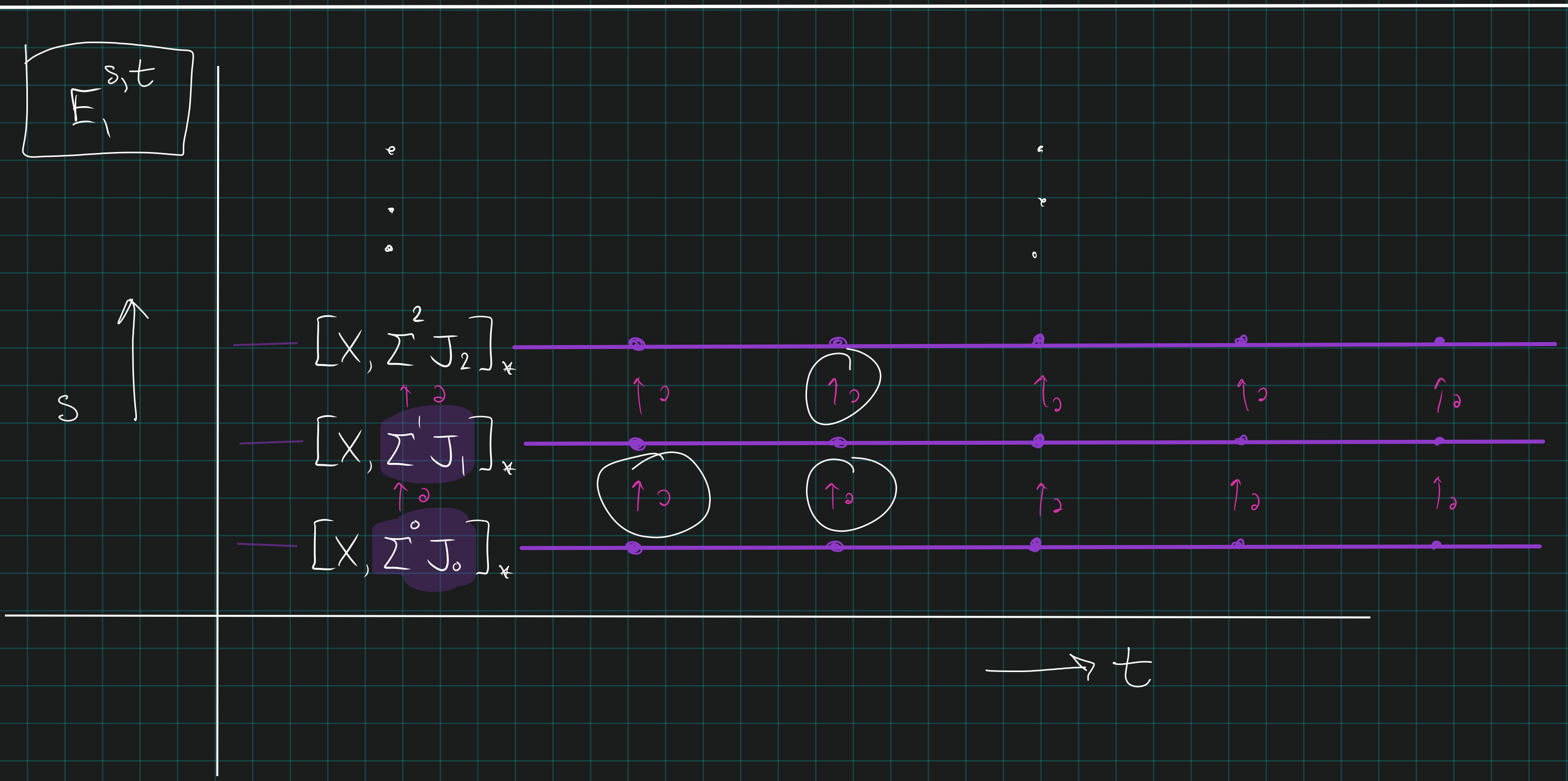
Constructing the AdSS: E_1 & Its Differential

- Use the exact couple to express E_1:

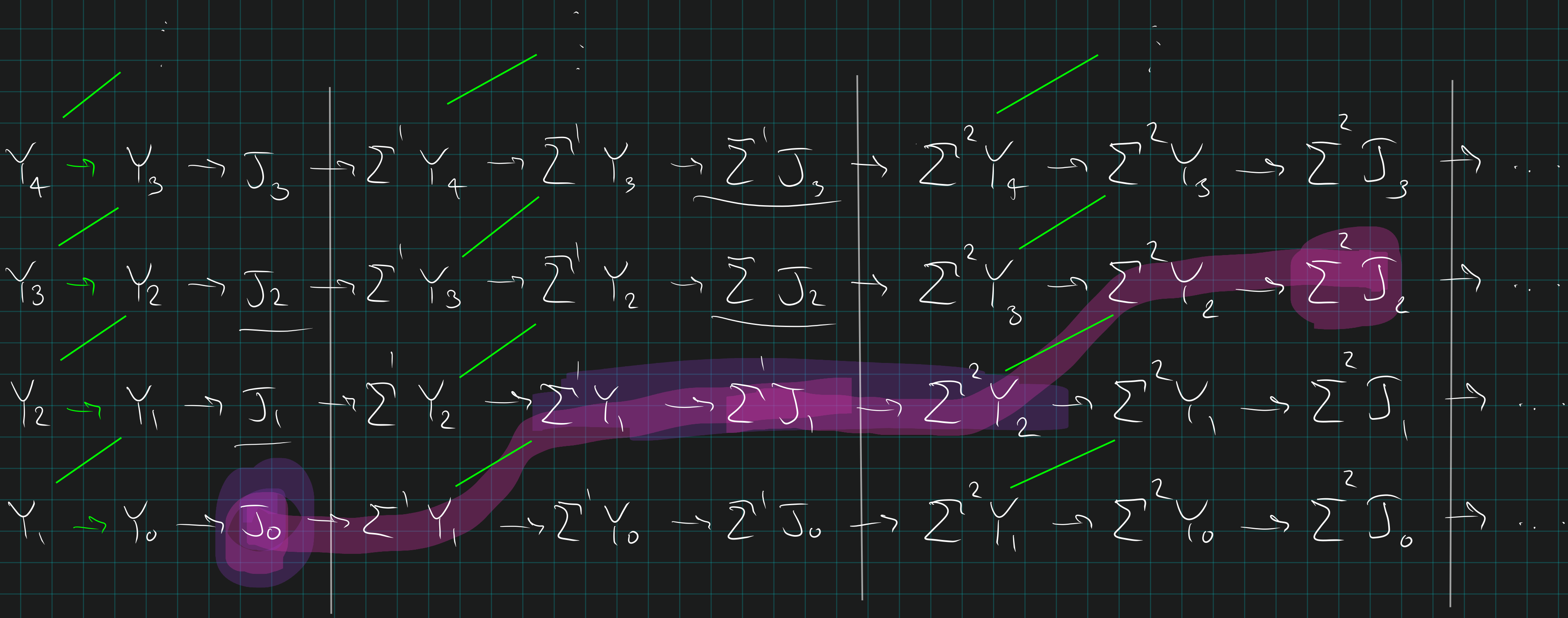
$$\begin{tikzcd}
\bigoplus [X, Y_n]_* \arrow[rr, "i"] & & \bigoplus_n [X, Y_n]_* \arrow[dl, "j"] \\
& \bigoplus_n [X, J_n]_* \arrow[ul, "K"] &
\end{tikzcd}$$

Maps isomorphically to $\mathrm{Hom}^s_A(H^*J_n, H^*X)$

$$\Rightarrow E_1^{s,t} = [X, \Sigma^s J_s]_t = [\Sigma^t X, \Sigma^s J_s] = \underbrace{[X, J_n]_{t-s}}$$



What is the differential ∂? Splice cofiber sequences:



- Yields maps $\partial_n : J_n \to \Sigma^{n+1} J_{n+1}$ $\forall n$ Apply $[X, \cdot]$

- Already have $\partial = 0$ in SHC:

$$\begin{tikzcd}
\underbrace{J_n \to \Sigma^1 Y_{n+1}} \arrow[rr, bend left, "\partial"] & \underbrace{\Sigma^1 Y_{n+1} \to \Sigma^1 J_{n+1} \to \Sigma^2 Y_{n+2}} \arrow[rr, bend left, "\partial"] & \underbrace{\Sigma^2 Y_{n+2} \to \Sigma^2 J_{n+2}}
\end{tikzcd}$$

in one level

$\therefore \partial^2 \simeq 0$ is null homotopic.

$\boxed{\text{Define } \partial: \text{ apply } [X, -] \text{ to } \partial}$

Constructing the AdSS : E_2

- **Goal:** Identify E_2 with an Ext in $\mathcal{A}$.

$$Y = Y_0 \to J_0 \xrightarrow{\partial} \Sigma^1 J_1 \xrightarrow{\partial} \Sigma^2 J_2 \xrightarrow{\partial} \cdots$$

$\downarrow$ Apply contravariant $H^*(-)$

$$\begin{array}{ccccccccc}
H^*Y = H^*Y_0 & \leftarrow & H^*J_0 & \leftarrow & H^*\Sigma^1 J_1 & \leftarrow & H^*\Sigma^2 J_2 & \leftarrow & \cdots \\
\| & & \| & & \downarrow\wr & & \downarrow\wr & & \\
H^*Y & \leftarrow & H^*J_0 & \leftarrow & H^{*-1}J_1 & \leftarrow & H^{*-2}J_2 & \leftarrow & \cdots
\end{array}$$

$\underbrace{\hphantom{H^*Y \leftarrow H^*J_0 \leftarrow H^{*-1}J_1 \leftarrow H^{*-2}J_2 \leftarrow \cdots}}_{}$

A projective resolution of H^*Y in $\mathcal{A}$-Mod

$$\begin{array}{c|c}
E_1^{s,t} & \\
 & \mathrm{Hom}_{\mathcal{A}}(H^*\Sigma^{s+1}J_{s+1}, H^*X)_t \\
 & \uparrow \partial \\
s \uparrow & \mathrm{Hom}_{\mathcal{A}}(H^*\Sigma^{s}J_{s}, H^*X)_t \\
 & \uparrow \partial \\
 & \mathrm{Hom}_{\mathcal{A}}(H^*\Sigma^{s-1}J_{s-1}, H^*X)_t \\
\hline
 & \longrightarrow t
\end{array}$$

$$E_2^{s,t} = H_s\left(\mathrm{Hom}_{\mathcal{A}}(H^*\Sigma^n J_n, H^{*+t}X), \partial\right)$$

$$\Rightarrow \quad E_2^{s,t} = \mathrm{Ext}^{s,t}_{\mathcal{A}\text{-Mod}}(H^*Y, H^*X).$$

- Computing derived functors of

$$F(-) = \mathrm{Hom}_{\mathcal{A}}(-, H^*X)$$

on $H^*Y \in \mathcal{A}$-Mod

Constructing the AdSS: Remarks

- Convergence for $E = E(n)$ [Johnson-Wilson theories] discussed in Hovey-Strickland.

- MU or BP (= p-local MU) yields the AdNSS (Adams-Novikov).

- The Thom reduction $\Phi \in Sp(BP, H\mathbb{F}_p)$ induces a morphism in $Sseq(\mathcal{A})$

  $$\text{AdNSS} \longrightarrow \text{AdSS}$$

  ↳ Apply Φ to towers:

  A morphism has Adams filtration $\leq s$

  $\Rightarrow$ AdN filtration $\leq s$.

- Method works for any cohom theory E, yields (+ conditions)

  $$\underbrace{E_2^{s,t} = \operatorname{Ext}^{s,t}_{E^*(E)}(E^*(Y), E^*(X))}\Rightarrow \underbrace{[X, L_E Y]_{t-s}}$$

Example: 1-Line for $\mathbb{S}$

- Take $X = Y = \mathbb{S}$, $E = H\mathbb{F}_2$: $E_2^{s,t} = \mathrm{Ext}_{\mathcal{A}}^{s,t}(\mathbb{F}_2, \mathbb{F}_2) \Rightarrow \pi_{t-s}\mathbb{S} \otimes \mathbb{Z}_{\hat{2}}$.

- **Claim:** $\mathrm{Ext}_{\mathcal{A}}^{1,t} = \begin{cases} \mathbb{F}_2[h_i], & t = 2^i \\ 0, & \text{else}. \end{cases}$

- **Proof**

- $\xi \in \mathrm{Ext}_{\mathcal{A}}^{1,t}(\mathbb{F}_2, \mathbb{F}_2) \rightleftharpoons \xi : \left(0 \to \mathbb{F}_2[t] \to M \to \mathbb{F}_2[0] \to 0\right)$
  $\in \mathcal{A}\text{-Mod}$

<u>Notation</u>

$\mathbb{F}_2[s]$ = one copy of $\mathbb{F}_2$ in

deg s, gen'd over $\mathbb{F}_2$ by x_s.

- **Rmk:** $\mathcal{A}\text{-Mod} \xrightarrow{\text{Forget}} gr^{\mathbb{Z}}\,\mathbb{F}_2\text{-Mod}$ Every $\mathcal{A}$-Mod is an $\mathbb{F}_p$-Mod

$\mathbb{F}_2[0] \in gr^{\mathbb{Z}}\,\mathbb{F}_2\text{-Mod}$ Free $\Rightarrow \xi$ splits here

$\Rightarrow M \xrightarrow{\sim} \mathbb{F}_2[t] \oplus \mathbb{F}_2[0]$ as $\mathbb{F}_2$-mod

$\qquad\qquad \mathbb{F}_2\langle x_0, x_t\rangle$, $\mathrm{rank}_{\mathbb{F}_2} M = 2$

<u>But</u> this may not split in $\mathcal{A}$-Mod!

Example: 1-Line for $\mathbb{S}$

- There is a correspondance

$$\left\{\begin{array}{l} Q \in \mathcal{A} \\ \deg Q = t \\ Q \cdot x_0 = x_t \end{array}\right\} \rightleftharpoons \left\{\begin{array}{l} \xi :\ 0 \to \mathbb{F}_2[t] \to M \to \mathbb{F}_2[0] \to 0 \\ \quad \text{in } \mathcal{A}\text{-Mod} \end{array}\right\}$$

Rmk: This forces Q to be <u>indecomposable</u>

Why?

Suppose $Q = Q_1 \cdot Q_2$

$Q x_0 = Q_1 Q_2 x_0 = x_s$
$= Q_1 \, x_0'$
where $x_0' \neq x_s$

$$\Rightarrow \left\{\begin{array}{l} Q x_0 = x_s \\ Q_2 x_0 \neq x_s \end{array}\right. \Rightarrow Q_2 = x_s' \neq x_s$$

$\mathbb{F}_2 \langle x_0, x_s, x_s' \rangle \leq M$

$\Rightarrow \operatorname{rank}_{\mathbb{F}_2} M \geq 3.$ //

So $Q \in \{ Sq^{2^k} \}_{k \in \mathbb{Z}^{\geq 0}}$ indecomposables in $\mathcal{A}$!

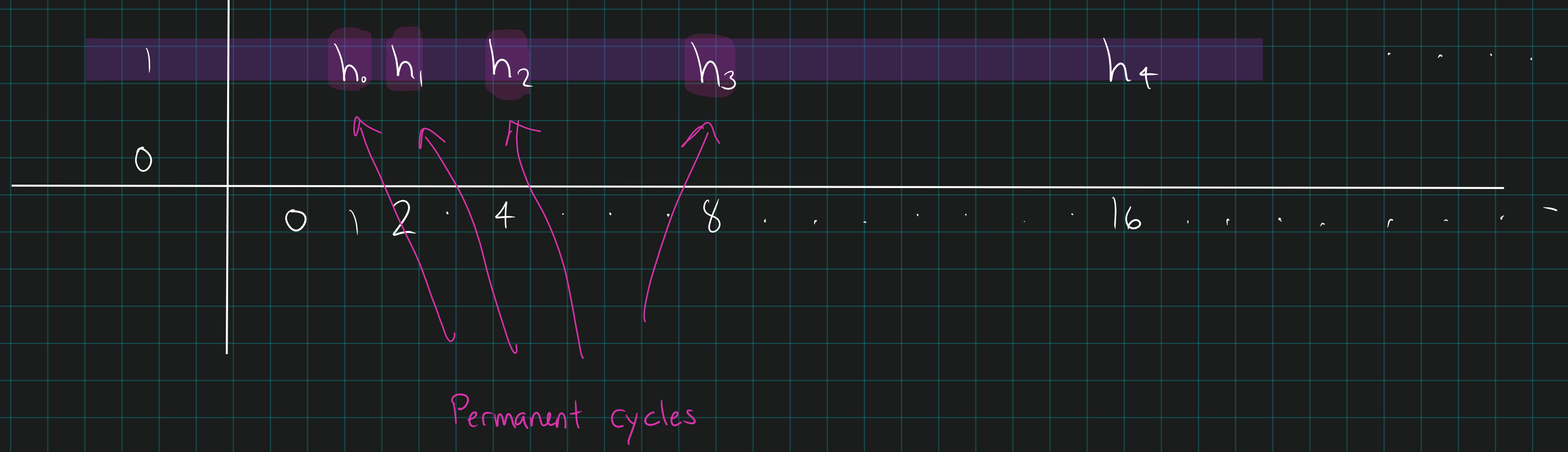
$$\Rightarrow \quad \operatorname{Ext}_{\mathcal{A}}^{1,t}(\mathbb{F}_2, \mathbb{F}_2) \rightleftharpoons \{ Sq^{2^k} \}_{k \in \mathbb{Z}^{\geq 0}}$$

$$h_k \rightleftharpoons Sq^{2^k}$$

$E_2^{1,t}$

	Bidegree (s,t)	Tot degree $(t-s)$	Hopf Elts
h_0	$\in (1,1)$	0	$2 \in \pi_0(\mathbb{S}) \otimes \hat{\mathbb{Z}}_2 = \hat{\mathbb{Z}}_2$
h_1	$\in (1,2)$	1	$\eta \in \pi_1(\mathbb{S}) \otimes \hat{\mathbb{Z}}_2 = C_2$
h_2	$\in (1,4)$	3	$\nu \in \pi_3(\mathbb{S}) \otimes \hat{\mathbb{Z}}_2 = C_8$
h_3	$\in (1,8)$	7	$\sigma \in \pi_7(\mathbb{S}) \otimes \hat{\mathbb{Z}}_2 = C_{16}$

Related to S^0, S^1, S^3, S^7 being parallelizable

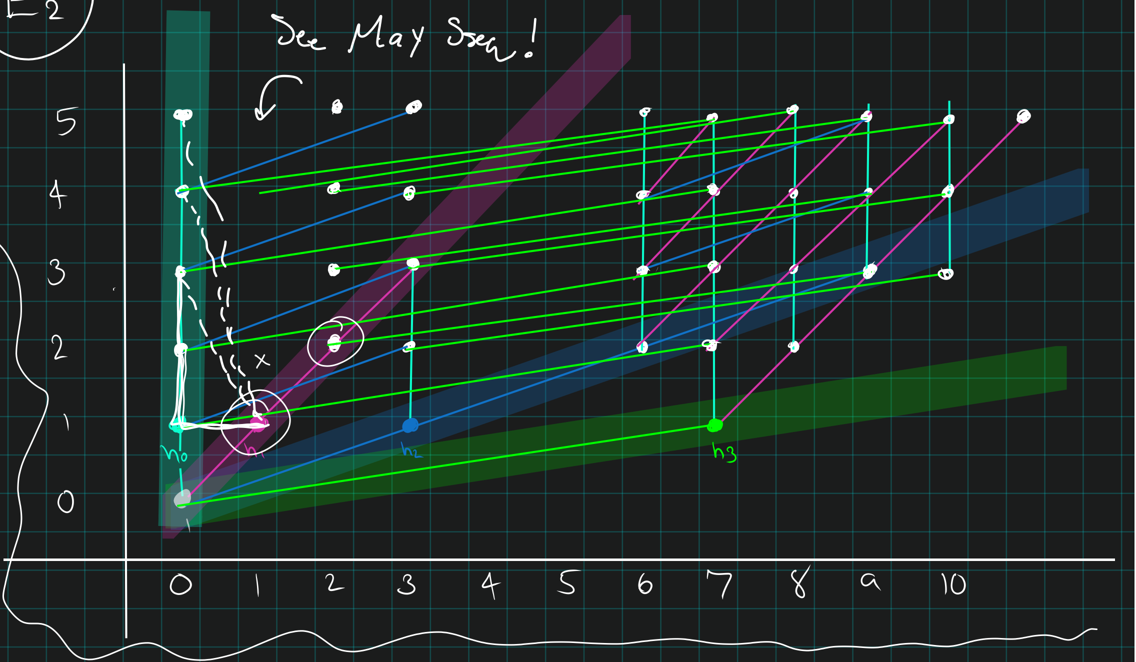


Example: 1-Line for $\mathbb{S}$

Remarks

- For $s \geq 2$, need Adem relns to write resolutions of $\mathbb{F}_p$ in $\mathcal{A}$-Mod
- Min. resolutions: few gens — detects by comparison thm
- Bar resolution: detects structure on $\pi_* \mathbb{S}$ (eg Toda brackets, product struct)
- Squares of Hopf elts: Kervaire elts? Check if h_n^2 are permanent cycles!

E_2



$n \leq 5$ permanent

$n = 6$ unknown

$n \geq 7$ not permanent cycles!

$\pi_2 \mathbb{S} = \mathbb{Z}/2$!