UGA GSS

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Why Derived Geometry?

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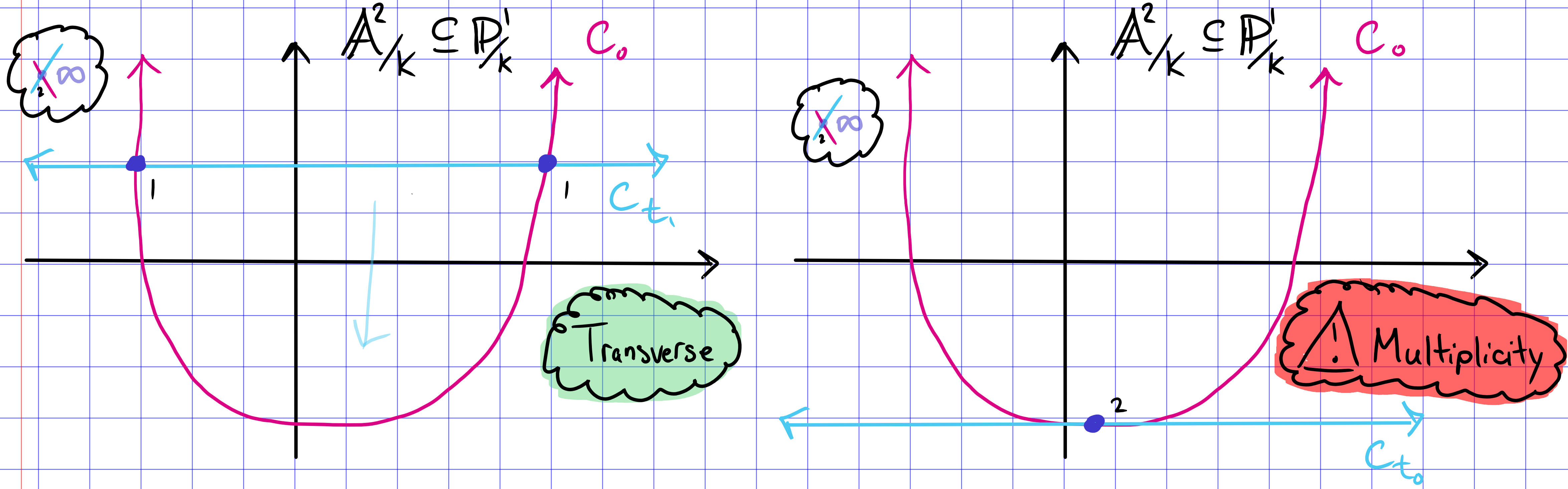
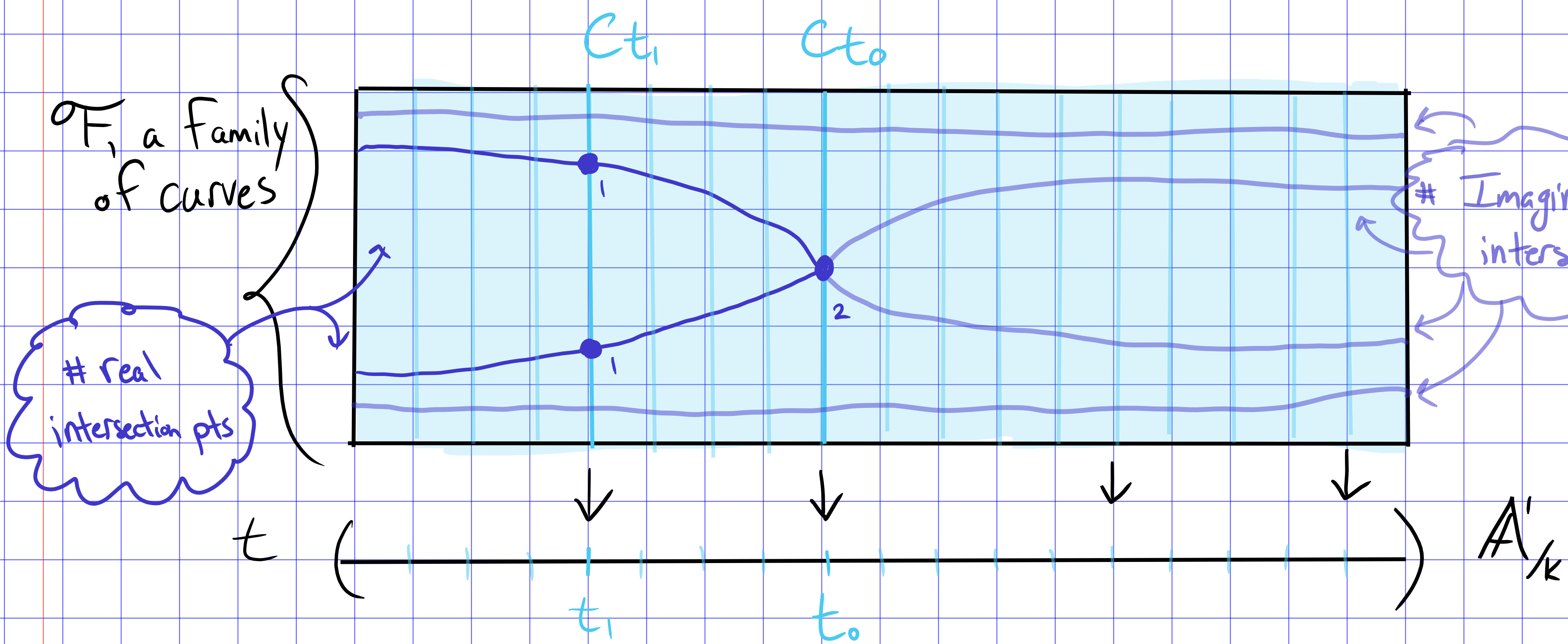
"Derived": Motivation from Intersection Theory





"Derived": Motivation from Intersection Theory

- Let $d_i := \deg C_i$, then

Transverse

$$C_1 \pitchfork C_2 \Rightarrow \#\{C_1 \cap C_2\} = d_1 \cdot d_2 \qquad \text{(Bezout)}$$

and for schemes,

$$\mu_p(\mathbb{P}^1; C_1, C_2) = \dim_{X(p)} \left(\mathcal{O}_{C_1,p} \otimes_{\mathcal{O}_{\mathbb{P}^1,p}} \mathcal{O}_{C_2,p} \right)$$

- For higher dimensions & non-transverse intersections, the fix:

<u>Serre's Intersection Formula</u>!

For $A, B \subseteq X$ smooth projective varieties,

$\dim A + \dim B = \dim X, \quad \dim(A \cap B) = 0,$

$$\mu_p(X; A, B) = \chi_q^{gr}\left(\pi_\bullet \left(\mathcal{O}_{A,p} \overset{\mathbb{L}}{\otimes}_{\mathcal{O}_{X,p}} \mathcal{O}_{B,p} \right) \right) \Big|_{q=1}$$

"Derived" (simplicial ring)

$$= \sum_{i \geq 0} (-1)^i \dim_{X(p)} \operatorname{Tor}_i^{\mathcal{O}_{X,p}} (\mathcal{O}_{A,p}, \mathcal{O}_{B,p}) \cdot q^i \Big|_{q=1}.$$

graded Euler characteristic

Moduli Problems

- Motivation from topology: consider the functor

$$\mathrm{Bun}_G(-) : \mathrm{hoTop} \longrightarrow \mathrm{Set}$$

$$X \mapsto \left\{ \begin{array}{c} G \to \mathcal{E} \\ \downarrow \\ X \end{array} \quad \text{Principal } G\text{-bundles} \right\} \Big/ \sim$$

where $\mathcal{E}_1 \sim \mathcal{E}_2 \Leftrightarrow$

$$\begin{tikzcd} \mathcal{E}_1 \arrow[rr, dashed, "\exists \varphi"] \arrow[dr] \arrow[rr, phantom, shift right=4, "\wr"] & & \mathcal{E}_2 \arrow[dl] \\ & X & \end{tikzcd} \quad (G\text{-equivariant})$$

This is representable by a space BG, i.e.

$$\mathrm{Bun}_G(-) \overset{\text{nat. iso}}{\cong} [-, BG] \text{ as sets}$$

Hty. homs

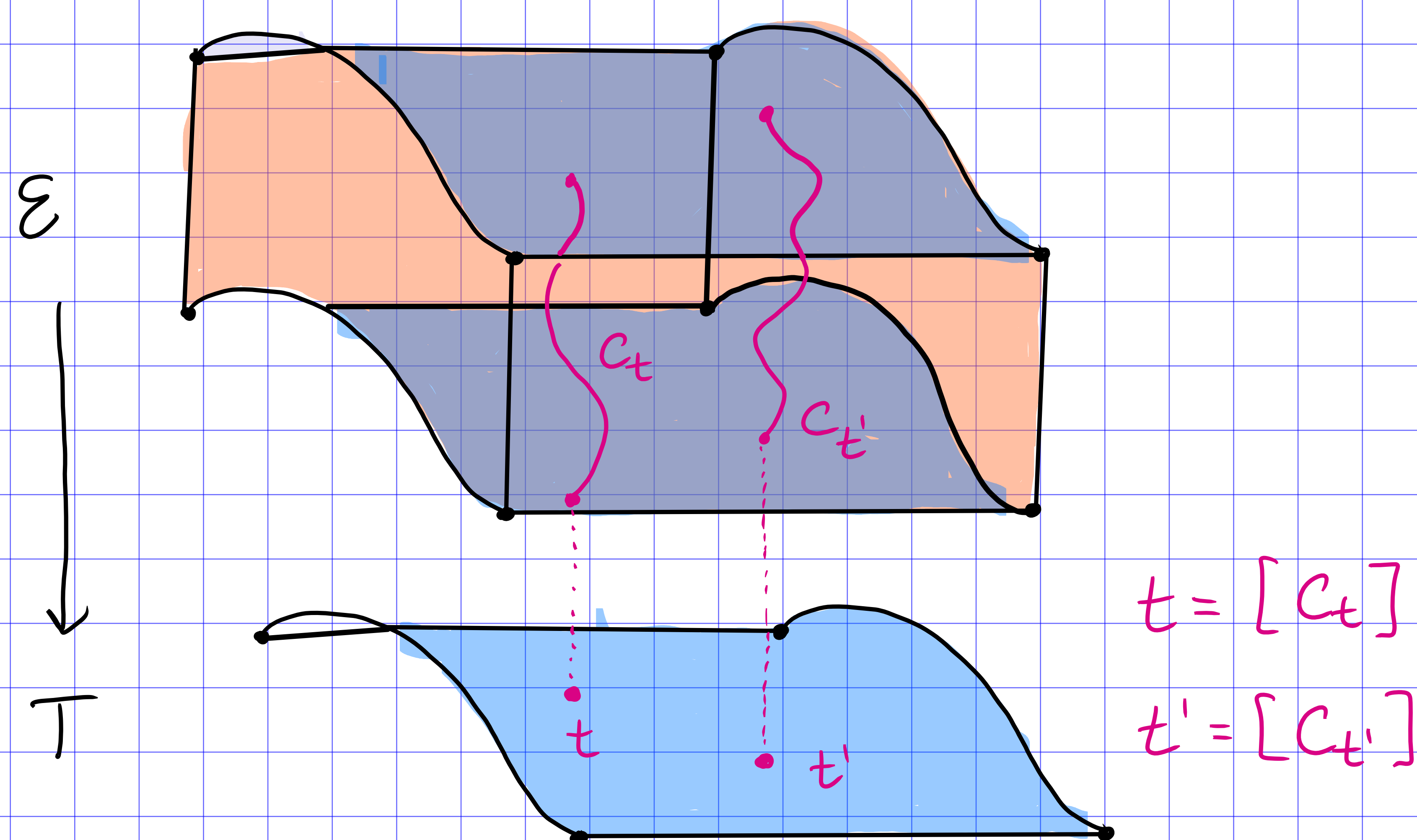
and all are pullbacks of a universal family:

$$\begin{tikzcd} \mathcal{E} \arrow[r, "\sim"] \arrow[dr] & f^*EG \arrow[r, dashed] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & EG \arrow[d] \\ & X \arrow[r, "f"] & BG \end{tikzcd}$$

Moduli Problems: Issues with Representability

- **Elliptic curve/k:** A pair (E,p) where E is a smooth projective curve, geometrically connected ($|E \times_k L|$ connected as a space for all L/k), $g(E) = 1$.

- **Family of curves/T:** $\mathcal{F} = \left\{ \begin{array}{c} \mathcal{E} \\ s \uparrow\downarrow \\ T \end{array} \begin{array}{l} \text{smooth} \\ \text{proper} \end{array} \middle| \begin{array}{l} (\mathcal{E}_t, s(t)) \text{ elliptic} \\ \text{over } k(t) \ \forall t \in T \end{array} \right\}$



Moduli Problems: Issues with Representability

- Attempt to define a functor

$$F : Sch_{/\mathbb{C}} \to Set$$

$$T \mapsto \left\{ \begin{array}{c} \mathcal{E} \\ s \uparrow \downarrow \\ T \end{array} \quad \text{families of ell. curves} \right\} \Big/ \sim$$

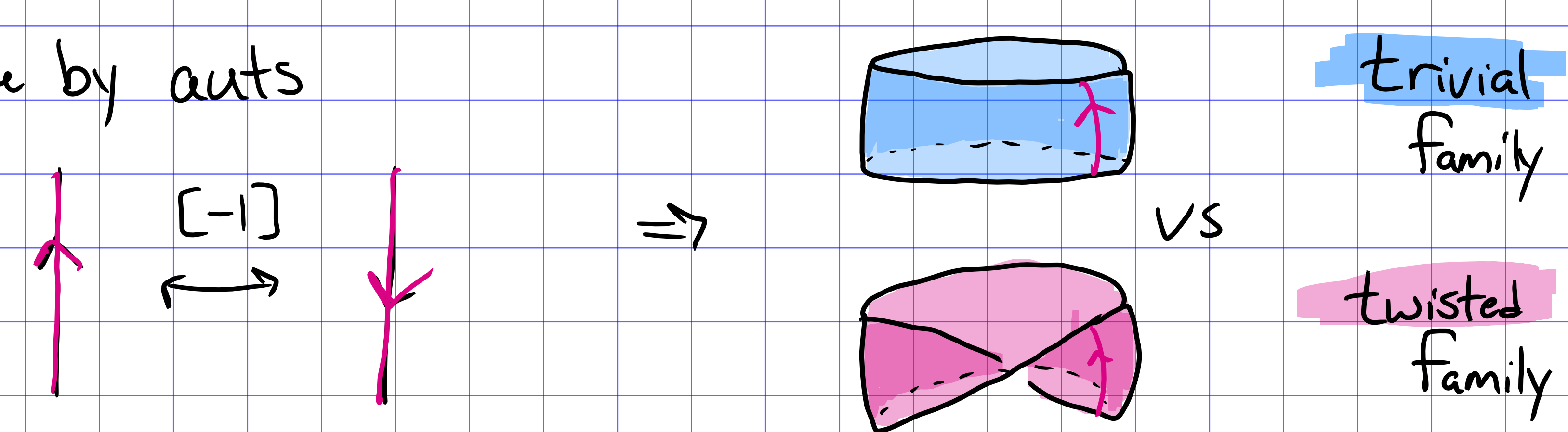
Problem: F is <u>not</u> representable; e.g. because curves have automorphisms $[-1]$ (see twisting construction).

There exist families $\mathcal{E} \in F(S)$ s.t. $\forall E$

$$\left\{ \begin{tikzcd} \mathcal{E} \arrow[r, "\not\cong"] \arrow[dr] & c^*E \arrow[r, dashed] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & E \arrow[d] \\ & S \arrow[r, "c"] & \operatorname{Spec} \mathbb{C} \end{tikzcd} \right\}$$

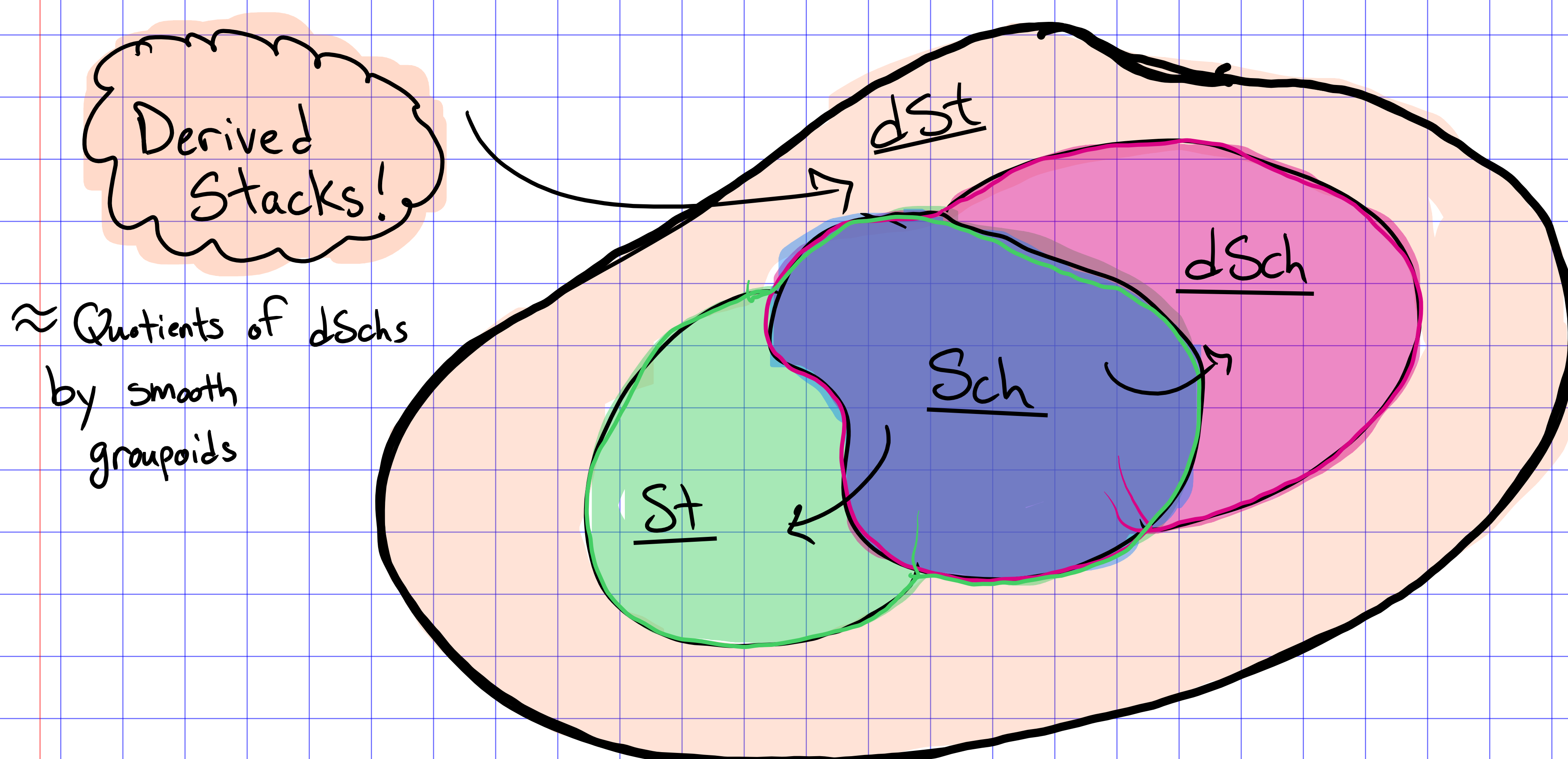
"$\mathcal{E}$ is a nontrivial family"

Idea: glue by auts



"Geometry": Stacks

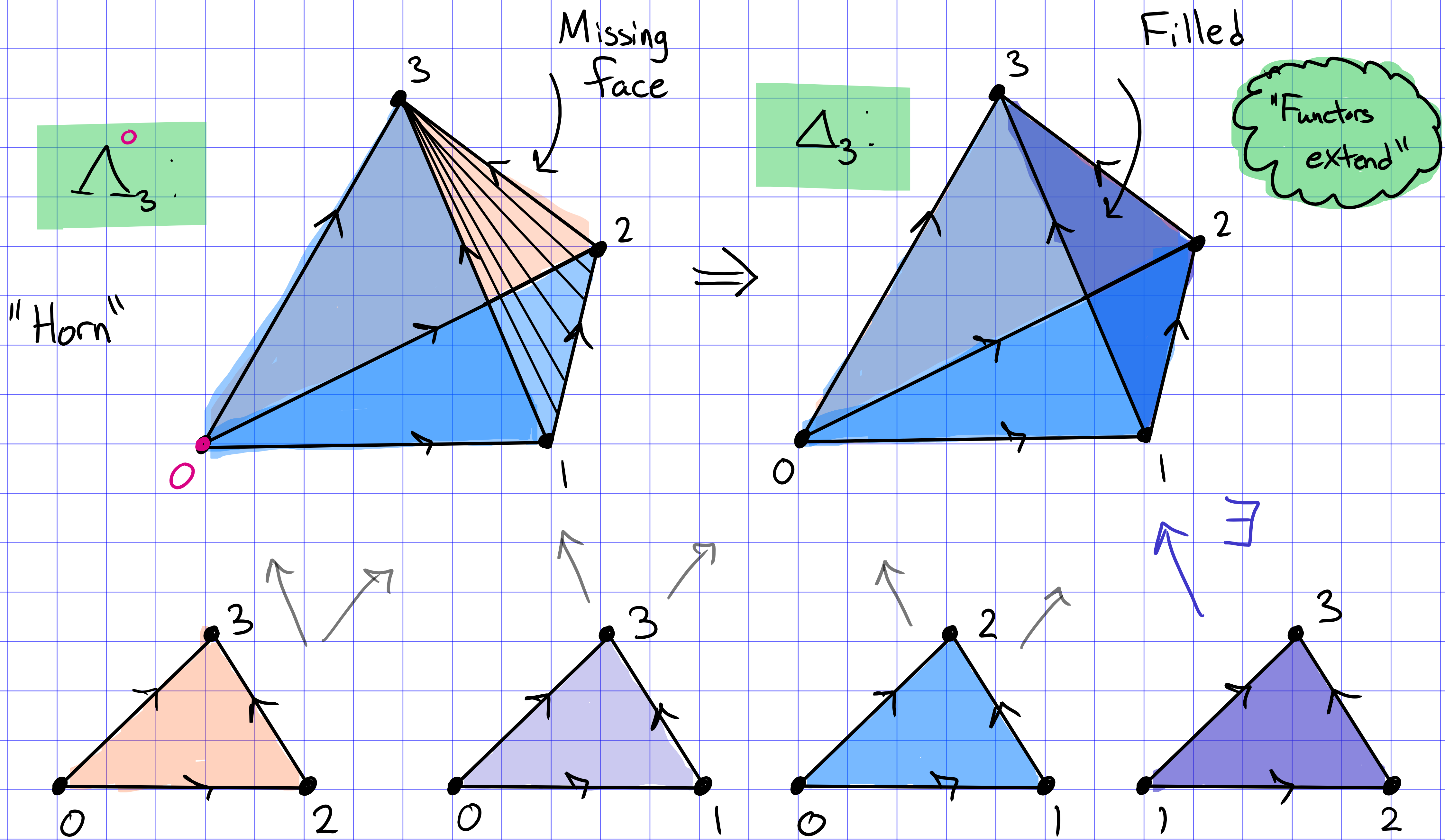
- **A fix:** enlarge Sch to another category:



$\boxed{\begin{array}{l} \text{dSch: } (X, \mathcal{O}_X^{\bullet}) \text{ where } \mathcal{O}_X \in \mathrm{Sh}(X; \mathcal{C}) \\ \text{where } \mathcal{C} = \text{simplicial CRings}, \mathbb{E}^{\infty}\text{-rings, cdgas}, \ldots \\ \text{and } (X, \pi_0 \mathcal{O}_X^{\bullet}) \in \mathrm{Sch}, \quad \pi_i \mathcal{O}_X^{\bullet} \in \mathrm{QCoh}(X; \pi_0 \mathcal{O}_X^{\bullet}) \ \forall i \end{array}}$

- $\mathrm{Sch} \to \mathrm{St}$: Closed under quotients by G-actions
- $\mathrm{Sch} \to \mathrm{dSch}$: Higher-order corrections for intersections
- $\mathrm{Sch} \to \mathrm{dSt}$: Both!

∞ - Categories



More precisely:

- **The simplex category** $\Delta = (\mathbb{Z}_{\geq 0}$, order-preserving functions$)$

$$[0] \to [1] \rightrightarrows [2] \rightrightarrows [3] \cdots\cdots$$

Generating morphisms
$$\begin{cases} \partial^n_i : [n] \hookrightarrow [n+1] & \text{Face morphisms, } 0 \leq i \leq n+1 \quad (\text{misses } i) \\ d^i_n : [n] \twoheadrightarrow [n-1] & \text{Degeneracy morphisms, } 0 \leq i \leq n \quad (\text{hits } i \text{ twice}) \end{cases}$$

∞- Cats

- **Simplicial objects** in $\mathcal{C}$: $\quad F \in \mathrm{Fun}(\Delta, \mathcal{C}) =: \underline{\Delta(\mathcal{C})}$

(Equiv: $\mathcal{C}$- valued copresheaves on Δ.)

E.g. $\quad \mathrm{Fun}(\Delta, \mathrm{Set}) =: \mathrm{sSet} = \underline{\Delta(\mathrm{Set})}$

Shorthand

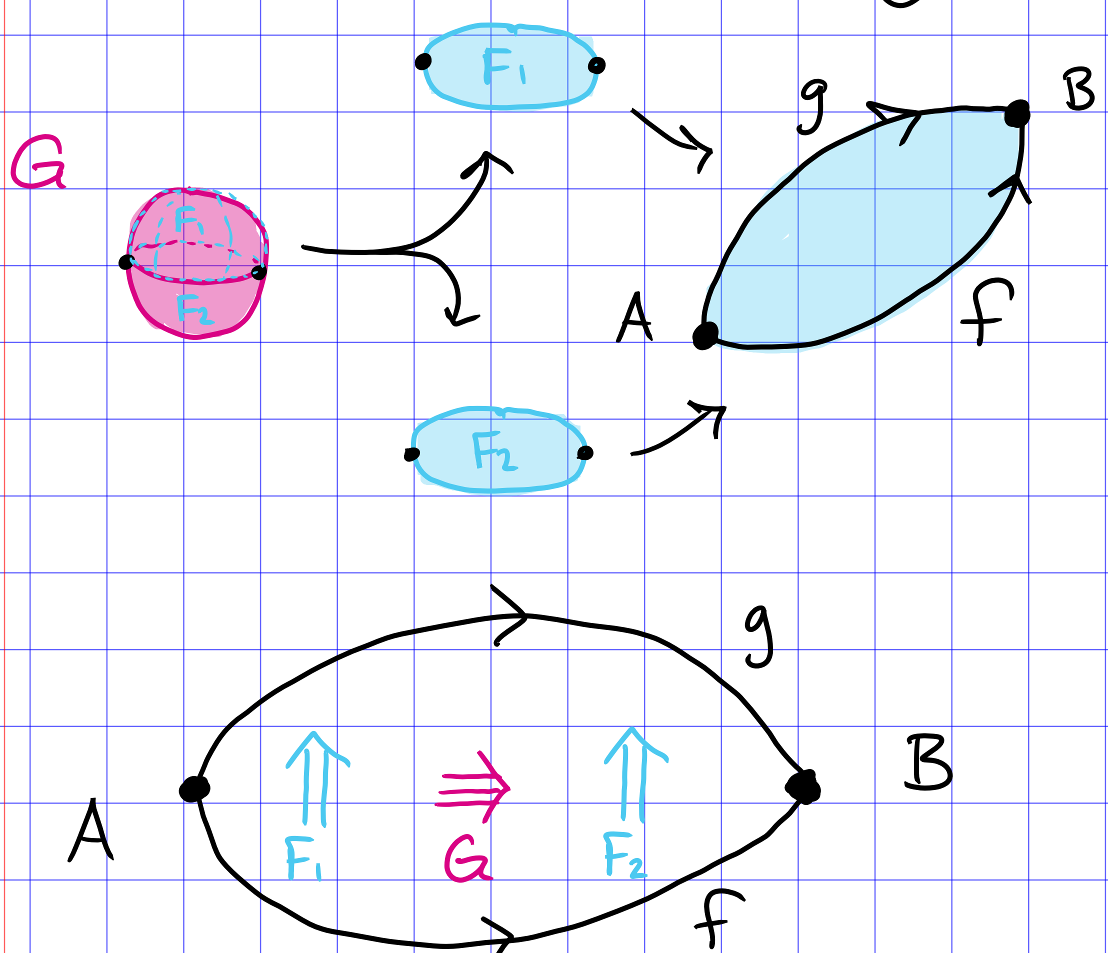
"Simplex - shaped diagrams in $\mathcal{C}$"

- **∞ - Category:**

$$\mathcal{C} \in \infty\text{-}\mathrm{Cat} \iff \mathcal{C} \in \Delta(\mathrm{Set}) \text{ and } \forall \quad \begin{tikzcd} \Delta^n \arrow[drr, dashed, "\exists"] & & \\ \Delta^n_i \arrow[u, hook] \arrow[rr] & & \mathcal{C} \end{tikzcd}$$

- **Idea:**

"Coherent cell attachment"



1 - morphisms

2 - morphisms

3 - morphisms

- Nerve $N(\mathcal{C}) \in \Delta(\mathrm{Set})$ for usual cats

$\Rightarrow \quad \mathrm{Cat} \hookrightarrow \infty\text{-}\mathrm{Cat}$

Derived / "Higher" Stacks

· **Idea:** Replace $X \in Sch$ with $h_X : Sch \to Set$
$$Sch \cong (k\text{-}Alg)^{op}.$$

$$\begin{tikzcd}
cAlg_{/k} \arrow[r, "h_X^{op}"] \arrow[dd] & Set \arrow[d, hook] \\
 & \infty\text{-}Grpd \simeq Spaces \arrow[d, "\mathcal{N}(-)"] \\
\Delta(cAlg_{/k}) \arrow[r, "\tilde{h}_X"] & \Delta(Sets)
\end{tikzcd}$$

] Schemes

] Homotopy hypothesis

] derived stacks

$\Rightarrow$ $dSt = \overset{\simeq}{Fun}\big(\Delta(cAlg_{/k}), \Delta(Sets)\big)$ + étale descent,

the functors preserving weak equivalences in a model structure + glue as étale sheaves.

(Needs to witness isomorphisms on all π_* !!)

Examples, Examples, Examples

- $\widehat{\mathcal{M}}_{g,n}(X,\beta)$, stable maps of curves into X

  a smooth proj. variety$/\mathbb{C}$, $\beta \in H^2(X;\mathbb{Z})$, $\beta = [C]$

  ↳ See Gromov-Witten theory

  ↳ $\widehat{\mathcal{M}}_{1,1}$ the derived moduli stack of ell. curves.

- $[X/G]$, quotient stack of G-orbits, $G \in \mathrm{AlgGrp}$

  ↳ E.g. $\mathbb{P}^n \simeq [\mathbb{A}^{n+1}/\mathbb{G}_m] \rightsquigarrow \mathbb{CP}^n \simeq [\mathbb{C}^n/\mathbb{C}^\times]$

- $\mathrm{Pic}(X/k)$, invertible sheaves

- $\mathrm{Map}(A,B)$ mapping stacks

  ↳ $\mathrm{Pic}(X/k) \simeq \mathrm{Map}(X, K(\mathbb{G}_m, 1))$

- $\mathrm{Bun}_G(X)$, algebraic G-bundles

- $BG \simeq \mathrm{Bun}_G(pt) = K(G,1)$

  ↳ principal G-bundles / torsors

- $K(G,n)$, Eilenberg-MacLane stacks

  ↳ "Étale sheaves with values in $\tau_{\leq n}$ Spaces."

  ↳ Define cohomology as $H^n(-;G) := \mathrm{Map}(-, K(G,n))$!

- $\mathrm{QCoh}(X)$, quasicoherent sheaves

- $S^n = K(\mathbb{Z}, n)$ spheres

- $\mathcal{L}X = \mathrm{Map}(S^1, X)$ loop stacks.

Tools in the Theory

- Intersection theory/formulas
- Six functor formalism
- ℓ-adic methods, trace formulas
- Riemann-Roch
- Algebraic de Rham theory, Hodge filtrations
- Motivic integration & cohomology
- Vanishing theorems ($\sim$ Serre/Kodaira vanishing)
- Virtual fundamental classes

Applications

Geometric Langlands conjecture:

For C a smooth proj. curve $/\mathbb{C}$,

$$\mathbb{D}\Big(\underbrace{\mathcal{D}\text{-Mod}\big(\mathrm{Vect}_n(C)\big)}_{\substack{\mathcal{D}\text{-modules (regular, holonomic)} \\ \text{on the stack of vector} \\ \text{bundles on } C}}\Big) \xrightarrow[\text{triang.}]{\simeq} \mathbb{D}_{Coh}\big(\mathrm{Loc}_n^{dR}(X)\big)$$

$$\wr\!\wr$$

$$\mathbb{D}_{Coh}\Big(\underbrace{\mathrm{Map}(C^{dR}, BGL_n)}_{\text{Rank } n \text{ flat bundles}}\Big)$$

Hot gossip as of '05: for this to have a chance of being true, need to replace Loc with a derived stack.

Thanks!