CRAAG

July 2021

Grothendieck '66,

$$\mathbb{C} \to \Omega^0 \to \Omega^1 \to \Omega^2 \to \cdots$$

"On the de Rham Cohomology of Algebraic Varieties"

Thanks, Daniel!!

Intro / Background

- $\Gamma(-) := \Gamma(X,-) : Sh_X \to \mathbb{Z}\text{-Mod}$
- $\mathcal{F} \in Sh/X$, $\quad H^*_{Sh}(X;\mathcal{F}) := R^*\Gamma(\mathcal{F})$
- $\mathcal{F}^\bullet \in Ch(Sh/X)$, $\underline{\mathbb{H}(X;\mathcal{F}^\bullet) := R^*\Gamma(\mathcal{F}^\bullet)}$

- $(X, \mathcal{O}_X) \hookrightarrow (X^{an}, \mathrm{Holo}(-,\mathbb{C})) \hookrightarrow (X^{sm}, C^\infty(-,\mathbb{C}))$

- The classical picture: for $X \in \mathrm{AffSch}/\mathbb{C}$, there are several de Rham theories applicable:

- $\mathcal{A}^p_{X^{sm}/\mathbb{R}}$ = Sheaf of C^∞/smooth p-forms

- $\Omega^p_{An\,X^{an}/\mathbb{C}}$ = Sheaf of holomorphic p-forms

- $\Omega^p_{Alg\,X/\mathbb{C}}$ = Sheaf of algebraic p-forms

Subject of today

$$\left.\begin{array}{l} \underline{\mathbb{C}} \to \mathcal{A}^0 \to \mathcal{A}^1 \to \cdots \to \mathcal{A}^n \to 0 \\ \underline{\mathbb{C}} \to \Omega^0_{An} \to \Omega^1_{An} \to \cdots \to \Omega^{2n}_{An} \to 0 \end{array}\right]$$

Exact by Poincaré lemmas

$$\left.\underline{\mathbb{C}} \to \Omega^0_{Alg} \to \Omega^1_{Alg} \to \cdots \to \Lambda^n\Omega^1_{Alg} \to 0\right]$$

Not exact

- Since the $\mathcal{A}^p$ are fine sheaves, $H^q(X^{sm};\mathcal{A}^p) = 0$ for $p > 0$

$$\mathbb{H}(X^{sm};\mathcal{A}^\bullet) = H^*[\Gamma(\mathcal{A}^0) \to \Gamma(\mathcal{A}^1) \to \cdots] := H^*_{dR}(X^{sm})$$

So we define

$$\left.\begin{array}{c} H_{dR}(X^{sm}) := \mathbb{H}(X^{sm};\mathcal{A}^\bullet) \\ \uparrow \\ H_{dR}(X^{an}) := \mathbb{H}(X^{an};\Omega^\bullet_{An}) \end{array}\right]$$

$$\uparrow$$

Iso, by this paper

$$H_{dR}(X) := \mathbb{H}(X;\Lambda^\bullet\Omega^1_{Alg})$$

Isomorphic by the de Rham theorem:

Exactness $\Rightarrow$ both are $\Gamma(-)$ acyclic resolutions of $\underline{\mathbb{C}}$, so

$\mathcal{A}^\bullet \xrightarrow{\sim} \underline{\mathbb{C}} \xrightarrow{\sim} \Omega^\bullet_{An\,X/\mathbb{C}}$ are quasi-isos in $Ch(Sh_X)$

$\Rightarrow$ isomorphic $\mathbb{H}^*$.

Intro / Background

- How to define hypercohomology

$$\begin{tikzcd}
 & \arrow[d] & \arrow[d] & \arrow[d] & \arrow[d] & \\
\arrow[r] & 0 \arrow[r] & \mathcal{I}^{0,1} \arrow[r] & \mathcal{I}^{1,1} \arrow[r] & \mathcal{I}^{2,1} \arrow[r] & {} \\
\arrow[r] & 0 \arrow[r] \arrow[u] & \mathcal{I}^{0,0} \arrow[r] \arrow[u] & \mathcal{I}^{1,0} \arrow[r] \arrow[u] & \mathcal{I}^{2,0} \arrow[r] \arrow[u] & {} \\
\arrow[r] & 0 \arrow[r] \arrow[u] & \mathcal{F}^{0} \arrow[r] \arrow[u] & \mathcal{F}^{1} \arrow[r] \arrow[u] & \mathcal{F}^{2} \arrow[r] \arrow[u] & {}
\end{tikzcd}$$

CE-Res.

$\mathcal{I}^{\bullet,\bullet} \xrightarrow{\sim} \mathcal{F}^{\bullet}$ quasi-iso in $Ch^2(Sh_{/X})$

Injective (or just $\Gamma(X,-)$ acyclic) resolution

$$\boxed{\mathbb{R}^*\Gamma(\mathcal{F}^\bullet) := H^* \, \mathrm{Tot}^{\Pi} \, \Gamma(\mathcal{I}^{\bullet,\bullet})}$$

Why this can reduce to $R^q\Gamma(-)$

Delete augmentation, apply $\Gamma(-)$.

Rmk: Can use Godement resolutions.

$$\begin{tikzcd}
\arrow[d] & \arrow[d] & \arrow[d] & \\
\Gamma(\mathcal{I}^{0,1}) \arrow[r, "\partial_h"] \arrow[u] & \Gamma(\mathcal{I}^{1,1}) \arrow[r, "\partial_h"] \arrow[u] & \Gamma(\mathcal{I}^{2,1}) \arrow[u] & {} \\
\Gamma(\mathcal{I}^{0,0}) \arrow[r, "\partial_h"'] \arrow[u, "\partial_v"] & \Gamma(\mathcal{I}^{1,0}) \arrow[r, "\partial_h"'] \arrow[u, "\partial_v"] & \Gamma(\mathcal{I}^{2,0}) \arrow[u, "\partial_v"] & {}
\end{tikzcd}$$

Thm: Each column is an injective resolution!

Take vertical homology, run sseq pages

E_1

$$\begin{tikzcd}
\arrow[d] & \arrow[d] & \arrow[d] & \\
H^1(X;\mathcal{F}_0) \arrow[r] \arrow[u] & 0 \arrow[r] \arrow[u] & 0 \arrow[u] & {} \\
H^0(X;\mathcal{F}_0) \arrow[r] \arrow[u] & 0 \arrow[r] \arrow[u] & 0 \arrow[u] & {}
\end{tikzcd}$$

- For any $(C^{\bullet,\bullet}, \partial_v, \partial_h) \in gr^{\mathbb{Z}^{\times 2}} Ch(\mathcal{A})$, there are two canonical sseqs:

$${}^{I}E_2^{p,q} = H^p_{\partial_h}\left(H^q_{\partial_v}(C^{\bullet,\bullet})\right) \Rightarrow H^{p+q}\, \mathrm{Tot}^{\oplus} C^{\bullet,\bullet}$$

$${}^{II}E_2^{p,q} = H^q_{\partial_v}\left(H^p_{\partial_h}(C^{\bullet,\bullet})\right) \Rightarrow H^{p+q}\, \mathrm{Tot}^{\Pi} C^{\bullet,\bullet}$$

Filter by rows/columns, use sseq for a filtered complex on $(\mathrm{Tot}^{\Pi} C^{\bullet,\bullet}, \widetilde{\partial_v + \partial_h})$

- Set $C^{p,q} := \Gamma(X, \mathcal{I}^{p,q})$, use that $\mathcal{F}^\bullet \xrightarrow{W} \mathcal{G}^\bullet$ (quasi-iso) $\Rightarrow \mathbb{H}(X,\mathcal{F}^\bullet) \xrightarrow{\sim} \mathbb{H}(X,\mathcal{G}^\bullet)$

$\Rightarrow$ If $\mathcal{F}^\bullet$ is $\Gamma(-)$ acyclic, $\mathbb{H}(X;\mathcal{F}^\bullet) = R^q\Gamma(\mathcal{F}^\bullet) = H^q(\Gamma\mathcal{F}^0 \to \Gamma\mathcal{F}^1 \to \cdots)$

- Def: $\Omega^1_{R/k}{}_{\mathrm{Alg}} := R\langle dr \mid r \in R \rangle \Big/ \left\langle \begin{matrix} d(r+s) = dr + ds \\ d(rs) = dr\cdot s + r\cdot ds \\ d(k) = 0 \end{matrix} \right\rangle$

$$\Omega^\bullet_{R/k}{}_{\mathrm{Alg}} := R \to \Omega^1 \xrightarrow{\partial} \Lambda^2\Omega^1 \xrightarrow{\partial} \Lambda^3\Omega^1 \xrightarrow{\partial} \cdots$$

$$\partial : \Lambda^n\Omega^1 \to \Lambda^{n+1}\Omega^1$$

$$f \Lambda dr_i \mapsto df \wedge (\Lambda dr_i)$$

For Schemes: $\Omega_{A/B} = \Delta^*(\mathcal{I}/\mathcal{I}^2)$ ideal sheaves for $\mathrm{im}(\Delta : X \to X \times_Y X)$

Eg: $R = K[x_1, \cdots, x_n] \Rightarrow \Omega^1_{R/k} = R[dx_1, \cdots, dx_n]$

$X = \mathbb{A}^n_Y \Rightarrow \Omega^1_{X/Y} = \mathcal{O}_X \langle \underbrace{dx_1, \cdots, dx_n}_{} \rangle$

$\hookrightarrow \in \Gamma(\mathcal{O}_X)$, x_i affine coords for $\mathbb{A}^n_Y$

The Paper

⚠ Reductions! All implications will run backward

(pre EGA: prescheme = scheme, scheme = sep. scheme ← $\Delta : X \to X^{\times 2}$ closed imm.)

quotient of $k[X] \in k\text{-Alg}$

Thm 1: $\boxed{\mathbb{H}(X; \Omega^\bullet_{\mathrm{Alg}\, X/\mathbb{C}}) \xrightarrow{\sim} H^*_{\mathrm{Sing}}(X^{an}; \mathbb{C})}$ When $X \in \mathrm{AffSch}^{\mathrm{Alg}}/\mathbb{C}$ is regular. (nonsingular)

Thm 1': $\boxed{\mathbb{H}(X; \Omega^\bullet_{\mathrm{Alg}\, X/\mathbb{C}}) \underset{\psi^*}{\xrightarrow{\sim}} \mathbb{H}(X^{an}; \Omega^\bullet_{\mathrm{Hol}\, X/\mathbb{C}})}$ is an iso for $X \in \mathrm{preSch}$, locally f.t., smooth $/\mathbb{C}$ (where ψ is a morphism of spectral sequences)

Rmk: If X is a smooth projective variety, (1) holds:

- $\mathbb{C}^\bullet \xrightarrow{w} \Omega^\bullet_{\mathrm{An}\, X/\mathbb{C}} \Rightarrow \mathbb{H}(X^{an}, \mathbb{C}) \xrightarrow{\sim} \mathbb{H}(X^{an}; \Omega^\bullet_{\mathrm{An}\, X/\mathbb{C}}) = \mathbb{H}^{an}$
- $E_{1\,\mathrm{An}}^{p,q} = H^p(X^{an}; \Omega^q_{\mathrm{An}}) \Rightarrow \mathbb{H}^{an} := \mathbb{H}(X^{an}, \Omega^\bullet_{\mathrm{An}\, X/\mathbb{C}})$

  $E_{1\,\mathrm{Alg}}^{p,q} = H^p(X; \Omega^q_{\mathrm{Alg}}) \Rightarrow \mathbb{H}^{alg} := \mathbb{H}(X^{an}, \Omega^\bullet_{\mathrm{Alg}\, X/\mathbb{C}})$
- $E_{1\,\mathrm{Alg}}^{\bullet,\bullet} \xrightarrow{\sim} E_{1\,\mathrm{An}}^{\bullet,\bullet}$ by GAGA, inducing $E_{\infty\,\mathrm{Alg}}^{\bullet,\bullet} \xrightarrow{\sim} E_{\infty\,\mathrm{An}}^{\bullet,\bullet} \Rightarrow \mathbb{H}^{an} \xrightarrow{\sim} \mathbb{H}^{alg}$
- Now apply $H^*_{\mathrm{Sing}}(X^{an}; \mathbb{C}) \cong \underbrace{\mathbb{R}^*\Gamma(X^{an}; \mathbb{C})}_{\text{sheaf cohom}} \cong \mathbb{H}(X^{an}; \mathbb{C}^\bullet) \cong \mathbb{H}^{an} \cong \mathbb{H}^{alg}$. ∎

Rmk (Groth) A similar argument can be used when X is complete ($X \times Y \xrightarrow{\pi_2} Y$ is closed)

Rmk (Reducing 1' to 1)

Step 1) Prove on affines

- Suppose X is affine, then consider
  - $\mathbb{H}(X; \Omega^\bullet_{\mathrm{Alg}\, X/\mathbb{C}})$ "(de Rham) hypercohomology"
  - $H^\bullet(X; \Omega^\bullet_{\mathrm{Alg}\, X/\mathbb{C}})$ "Hodge cohomology"
- Hodge to de Rham sseq:

$$E_1^{p,q} = \underbrace{\boxed{H^q(X; \Omega^p_{\mathrm{Alg}\, X/\mathbb{C}})}}_{\text{Hodge}} \Rightarrow \underbrace{\boxed{\mathbb{H}(X; \Omega^\bullet_{\mathrm{Alg}\, X/\mathbb{C}})}}_{\text{de Rham}}$$

- X affine $\Rightarrow \mathbb{H}(X; \Omega^\bullet_{\mathrm{Alg}\, X/\mathbb{C}}) = H^*(\Gamma(X; \Omega^\bullet_{\mathrm{Alg}\, X/\mathbb{C}}))$ Why: $\Omega^\bullet_{\mathrm{Alg}\, X/\mathbb{C}}$ is an exact complex

  $= H^*(\Gamma(\Omega^1_{\mathrm{Alg}}) \longrightarrow \Gamma(\Omega^2_{\mathrm{Alg}}) \longrightarrow \cdots)$

Step 2) Globalize

- Take an open cover $\mathcal{U} \rightrightarrows X$ (not nec. affine)

$$\begin{aligned} E_{2\,\mathrm{Alg}}^{p,q} &= \check{H}^p(\mathcal{U}; \mathcal{H}^q) \Rightarrow \mathbb{H}(X; \Omega^\bullet_{\mathrm{Alg}\, X/\mathbb{C}}) \\ &\Big\downarrow 2\ \varphi \quad \underline{\text{WTS}} \qquad\qquad \Big\downarrow \therefore \\ E_{2\,\mathrm{An}}^{p,q} &= \check{H}(\mathcal{U}^{an}; \mathcal{H}^q) \Rightarrow \mathbb{H}(X^{an}; \Omega^\bullet_{\mathrm{An}\, X/\mathbb{C}}) \end{aligned}$$

$\mathcal{H}^q : \mathrm{Op}(X) \to R\text{-Mod}$

$\mathcal{U} \mapsto H^q(\mathcal{U})$

Groth: "prescheme" ?

φ is an iso if: 1' holds when $X = \bigcap_{i_k \in I} U_{i_k}$ is a scheme*

1': $\mathbb{H}^{an}_{dR} \cong \mathbb{H}^{alg}_{dR}$

$\Rightarrow X \subseteq Y \in \mathrm{AffSch}$ since we're assuming locally f.t.

$\Rightarrow X$ is separated

$\Rightarrow U_{i_k}$ are affine

So X = intersection of affine opens + nonsingular

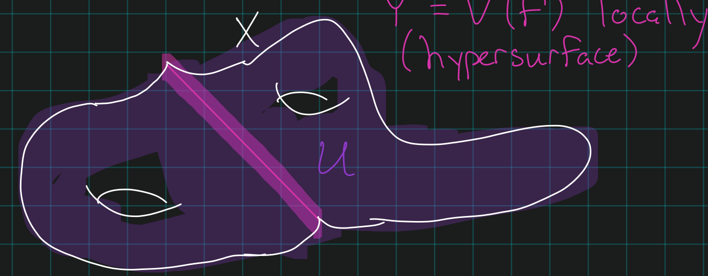
= nonsingular affine

$\rightsquigarrow$ Reduced to Thm 1.

The Paper

Thm 2

- $X \in \mathrm{Top}$ reduced, $\mathbb{C}$-an space
- $Y \subseteq X$ closed $\mathbb{C}$-an (locally one eqn)
- $U := X \setminus Y \subseteq X$ dense & nonsingular
- $\Omega^\bullet_{X/\mathbb{C}}(*Y) :=$ Extensions of $\Omega^\bullet_{U/\mathbb{C}}$ to $\Omega^\bullet_{X/\mathbb{C}}$ with at worst polar singularities along Y



$\hookrightarrow V \mapsto \left\{ \omega \in f_* \Omega^\bullet_{U,\,\mathrm{Hol}}(V \cap U) \;\middle|\; \omega = \alpha|_V \text{ where } \alpha \text{ is a meromorphic Kähler dif. form on } V \right\} \leq f_*(\Omega^\bullet_U) \in Sh_X$ (subsheaf)

$\forall y \in V \cap Y \; \exists \tilde{V} \subseteq \tilde{V}_y \cap Y$ open $\ni y$,
$\Omega^\bullet_U$ can be extended to a coherent sheaf $\mathcal{F}$ on $\tilde{V}_y$,
$\exists \alpha \in \Gamma(\tilde{V}, \mathcal{F})$ where $\omega|_{\tilde{V} \cap U} = \alpha / f^n \big|_{\tilde{V} \cap U}$

Sheaf of germs of q-forms, holo on $X \setminus Y$, polar sing. along Y

Then there is a canonical iso

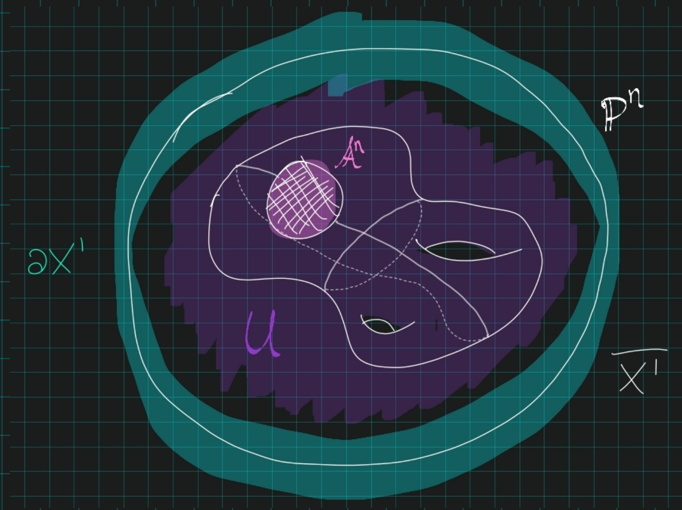
$$\mathcal{H}^q(\Omega^\bullet_{X/\mathbb{C}}(*Y)) \xrightarrow{\sim} R^q f_*(\underline{\mathbb{C}_U}) \quad \text{where } U \overset{f}{\hookrightarrow} X$$

Thm 2':

- X as above
- $H^p(X; \Omega^q_X(*Y)) = 0$ for $p > 0$, all q
- Either
  a) X is projective, $Y = \mathrm{supp}(D)$, $D \in \mathrm{Div}(X)$ ample, positive
  b) X is affine.

Then there is an isomorphism $H^q(\Gamma(X; \Omega^\bullet_{X/\mathbb{C},\,\mathrm{Alg}}(*Y))) \xrightarrow{\sim} H^q_{\mathrm{sing}}(U; \mathbb{C})$

$\text{Thm } 2' \Rightarrow \text{Thm } 1$



- Idea:

  - Set $\mathcal{U} := X_1$ as in thm 1 ($X_1 \in \mathrm{AffSch}^{\mathrm{Alg}}_{/\mathbb{C}}$ regular)

  - Embed $\mathcal{U} \hookrightarrow X := \overline{\mathcal{U}}$ a projective closure

  - Apply 2'.a:

    - X is projective? ✓

    - $Y := X \setminus \mathcal{U}$ is the support of a positive ample divisor?

      True: $\overline{\mathcal{U}} = \mathcal{U} \cup D$. $\mathcal{U} \hookrightarrow \mathbb{A}^n \Rightarrow \overline{\mathcal{U}} \hookrightarrow \mathbb{P}^n$, $D := \underset{= \partial\mathcal{U}}{\overline{\mathcal{U}} \setminus \mathcal{U}} \hookrightarrow \underbrace{\mathbb{P}^{n-1}}$ "hyperplane at ∞"

      $\overline{\mathcal{U}} = \mathcal{U} \cup \partial\mathcal{U}$, $\partial\mathcal{U} = \overline{\mathcal{U}} \cap \underset{= \{x_0 = 0\}}{H_0}$ the hyperplane section

      Hyperplane sections are ample divisors ✓

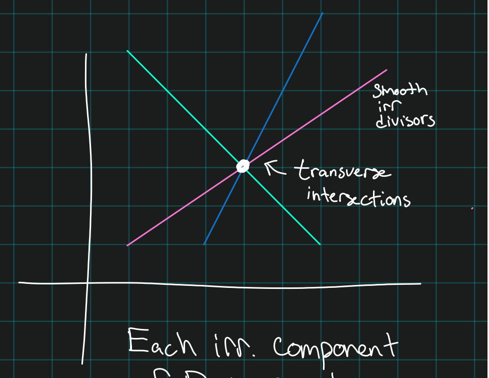
$$\Rightarrow \boxed{H^q\left(\Gamma(\overline{X}_1; \Omega^\bullet_{X/\mathbb{C}}(*\partial X'))\right) \xrightarrow{\sim} H^q_{\mathrm{Sing}}(X; \mathbb{C})}$$

WTS: $$\boxed{\mathbb{H}(X; \Omega^\bullet_{\substack{X/\mathbb{C} \\ \mathrm{Alg}}}) \xrightarrow{\sim} H^*_{\mathrm{Sing}}(X^{\mathrm{an}}; \mathbb{C})}$$

Perhaps $H^q\left(\Gamma(\overline{X}_1; \Omega^\bullet_{X/\mathbb{C}}(*\partial X'))\right) \cong \mathbb{H}(X; \Omega^\bullet_{\substack{X/\mathbb{C} \\ \mathrm{Alg}}})$?

Proof of Thm 2

D a SNC: Locally $\prod x_i = 0$, looks like hyperplane intersection



Each irr. component of D is regular

- Apply resolution of singularities to get $\hat{X} \xrightarrow{g} X$ projective & birational
  - ↳ $\hat{X}$ nonsingular, $\hat{Y} := g^{-1}(Y)$ a SNC divisor
  - ↳ Define $\mathcal{U} = X \setminus Y$, $\hat{\mathcal{U}} := g^{-1}(\mathcal{U})$, note $\hat{\mathcal{U}} \underset{g}{\overset{\text{iso}}{\xrightarrow{\sim}}} \mathcal{U}$

$$\begin{array}{ccc} (\hat{X}, & \hat{Y}, & \mathcal{U}) \\ \downarrow g & \downarrow g & \updownarrow \wr \\ (X & Y & \mathcal{U}) \end{array}$$

> ## Res of singularities (Hironaka)
>
> For X a reduced variety, a regular irreducible $\hat{X}$ with a birational proper morphism $\hat{X} \xrightarrow{f} X$ inducing $f^{-1}(X^{reg}) \xrightarrow{\sim} X^{reg}$ (Note $X^{reg} \subseteq X$ is Zariski-dense)

- Take injective resolutions:

$$\left.\begin{array}{l} \mathcal{L} := \underline{\mathbb{C}_{\mathcal{U}}} \to \mathcal{L}^1 \to \mathcal{L}^2 \to \cdots \\ \hat{\mathcal{L}} := \underline{\mathbb{C}_{\hat{\mathcal{U}}}} \to \hat{\mathcal{L}}^1 \to \hat{\mathcal{L}}^2 \to \cdots \end{array}\right\} \quad \begin{array}{l} f: \mathcal{U} \hookrightarrow X \\ \hat{f}: \hat{\mathcal{U}} \hookrightarrow \hat{X} \end{array}$$

- STS there are morphisms

$$\left.\begin{array}{ll} (a) & \Omega^\bullet_X(*Y) \longrightarrow f_* \mathcal{L}^\bullet \\ (b) & \Omega^\bullet_{\hat{X}}(*\hat{Y}) \longrightarrow \hat{f}_* \hat{\mathcal{L}}^\bullet \end{array}\right\} \text{ which will yield morphisms } \left\{\begin{array}{ll} \mathcal{H}^i(\Omega^\bullet_X(*Y)) \longrightarrow R^i f_*(\underline{\mathbb{C}_{\mathcal{U}}}) & (a') \\ \mathcal{H}^i(\Omega^\bullet_{\hat{X}}(*\hat{Y})) \longrightarrow R^i \hat{f}_*(\underline{\mathbb{C}_{\hat{\mathcal{U}}}}) & (b') \end{array}\right.$$

- STS (b), since $g: \hat{\mathcal{U}} \xrightarrow{\sim} \mathcal{U}$ an iso implies

Apply $g_*(-)$

Take homology

$$\begin{tikzcd} \Omega^\bullet_{\hat{X}}(*\hat{Y}) \arrow[r, "\sim"] \arrow[d] & \hat{f}_* \hat{\mathcal{L}}^\bullet \arrow[d] \\ g_*(\Omega^\bullet_{\hat{X}}(*\hat{Y})) \arrow[r, "\therefore \sim"] \arrow[d, "\wr"] & g_*(\hat{f}_* \hat{\mathcal{L}}^\bullet) \arrow[d, "\wr"] \\ \Omega^\bullet_X(*Y) \arrow[r, "\therefore \sim"] \arrow[d] & f_* \mathcal{L}^\bullet \arrow[d] \\ \mathcal{H}^i(\Omega^\bullet_X(*Y)) \arrow[r, "\therefore \sim"] & R^i f_*(\underline{\mathbb{C}_{\mathcal{U}}}) \end{tikzcd}$$

Not necessarily isos

(b) WTS this is an iso

Functoriality of $g_*(-)$ preserves isos

(a) Follows immediately, square of isos

+ ε argument

Then we win!

Proof of Thm 2 — Derived pushforwards vanish

- WTS: $\Omega^\bullet_{\hat{X}}(*\hat{Y}) \xrightarrow{\sim} \hat{f}_* \hat{\mathcal{L}}^\bullet$

- **Claim:** $\underbrace{R^q g_*(\Omega^p_{\hat{X}}(*\hat{Y}))=0}_{①}$, $\underbrace{R^q g_*(\hat{f}_*\hat{\mathcal{L}}^p)=0}_{②}$ for $q>0$, $\forall p$ (Right-derive both sides)

- ② : $\hat{f}_*\hat{\mathcal{L}}^p$ is a flasque sheaf ($U \hookrightarrow V \Rightarrow \mathcal{F}(V) \twoheadrightarrow \mathcal{F}(U)$.)

> **Stacks 20.12.5:** $(X,\mathcal{O}_X) \xrightarrow{g} (Y,\mathcal{O}_Y)$, $\mathcal{F} \in Sh_X(\mathcal{O}_X\text{-Mod})$ flasque
>
> $\Rightarrow R^q g_* \mathcal{F} = 0$ for $q>0$ ($\mathcal{F}$-acyclic for $F(-) := g_*(-)$).

- ①: More generally, for $\hat{\mathcal{E}} \in Coh(\hat{X})$, set $\hat{\mathcal{E}}(*\hat{Y}) :=$ Meromorphic sections of $\hat{\mathcal{E}}$ that are holomorphic on $\hat{U}$ $(:= \hat{X} \setminus \hat{Y})$

Claim: $R^q g_*(\hat{\mathcal{E}}(*\hat{Y})) = 0$ for $q>0$.

Pf: $\hat{\mathcal{E}}(*\hat{Y}) \xrightarrow{\sim} \varinjlim_n \hat{\mathcal{E}}_n$, each $\hat{\mathcal{E}}_n \in Coh(\hat{X})$

- Can take $\hat{\mathcal{E}}_n := \mathcal{H}om_{\mathcal{O}_X}(\hat{\mathcal{I}}^n, \hat{\mathcal{E}})$ where $\hat{\mathcal{I}}$ is a coherent sheaf of ideals defining $\hat{Y}$

- Use that $g: \hat{X} \to X$ is proper (by res. of singularities)

  $\Rightarrow R^q g_*(\varinjlim_n \mathcal{F}_n) \xrightarrow{\sim} \varinjlim_n R^q g_*(\mathcal{F}_n)$

> **Stacks 07TA:** $g: A \to B$ a quasi-compact & quasi-separated morphism of schemes, $\mathcal{F}_n \in QCoh(X)$
>
> $\Rightarrow R^q g_* \operatorname{colim}_n \mathcal{F}_n \xrightarrow{\sim} \operatorname{colim}_n R^q f_* \mathcal{F}_n$
>
> **Pf:** LHS: $(U \mapsto H^q_{sh}(f^{-1}(U); \operatorname{colim}_n \mathcal{F}_n))^{sh}$
>
> RHS: $(V \mapsto \operatorname{colim}_n H^q_{sh}(f^{-1}(V); \mathcal{F}_n))^{sh}$
>
> So check on affines: $\boxed{\operatorname{colim}_n H^q(U;\mathcal{F}_n) \xrightarrow[\forall q\geq 0]{\sim} H^q(U, \operatorname{colim}_n \mathcal{F}_n)}$
>
> Stacks 01FF
> Works for quasi-compact opens (+ conditions)

$\Rightarrow R^q g_*(\hat{\mathcal{E}}(*\hat{Y})) \xrightarrow{\sim} \varinjlim_n R^q g_*(\mathcal{E}_n)$

Take stalks

STS: $R^q g_*(\hat{\mathcal{E}}(*\hat{Y}))_x \xrightarrow{\sim} \varinjlim_n R^q g_*(\mathcal{E}_n)_x$

Proof of Thm 2

Isos on Stalks

WTS: $R^q g_*(\hat{\mathcal{E}}(*\hat{Y}))_x \xrightarrow{\sim} \varinjlim_n R^q g_*(\mathcal{E}_n)_x$

- Reduce to showing

$$H^q(\tilde{X}' ; \tilde{f}'_*(\tilde{\mathcal{E}}'|_{\tilde{U}'})) = 0 \qquad \forall q > 0$$

where
- $\tilde{X} = \mathrm{Spec}(\mathcal{O}_{X,x})$
- $\tilde{X}'/\tilde{X}$ defines $\hat{X}/X$ in a neighborhood of x
- $\tilde{\mathcal{E}}'_n \in \mathrm{Coh}(\tilde{X}')$ corresponds to $\hat{\mathcal{E}}_n$, $\tilde{\mathcal{E}}' := \mathrm{colim}_{\substack{\to \\ n}} \tilde{\mathcal{E}}'_n$?
- $\tilde{U}'$ corresponds to $\hat{U}$

- Why this is true:

$$\left.\begin{array}{l} \tilde{Y} \subseteq \tilde{X} \\ \tilde{Y}' \subseteq \tilde{X}' \end{array}\right\} \text{both defined by one equation}$$

$\Rightarrow \tilde{f}' : \tilde{U}' \hookrightarrow \tilde{X}'$ is affine

$$\Rightarrow H^q(\tilde{X}' ; \tilde{f}'_*(\tilde{\mathcal{E}}'|_{\tilde{U}'})) \xrightarrow{\sim} H^q(\tilde{U}' ; \tilde{\mathcal{E}}')$$

$$\Big\downarrow \wr\, \tilde{g}$$

$$\xrightarrow{\sim} H^q(\tilde{U} ; \tilde{\mathcal{E}}')$$

$= 0$ for $q > 0$ since

$\tilde{U}$ is affine

Proof of Thm 2 $\mathcal{E}$-Argument

- So we've established:

$$R^q g_*(\hat{\mathcal{E}}(*\hat{Y}))_x \xrightarrow{\sim} \varinjlim_n R^q g_*(\mathcal{E}_n)_x \quad \forall x \Rightarrow R^q g_*\left(\Omega^p_{\hat{X}}(*\hat{Y})\right) = 0, \quad R^q g_*(\hat{f}_* \hat{\mathcal{Z}}^p) = 0$$

$$\Rightarrow \Omega^\bullet_{\hat{X}}(*\hat{Y}) \xrightarrow{\sim} \hat{f}_* \hat{\mathcal{Z}}^\bullet$$

$$\text{WTS} \Rightarrow \mathcal{H}^q(\Omega^\bullet_X(*Y)) \xrightarrow{\therefore \text{iso}} R^q f_*(\underline{\mathbb{C}_U}) \qquad \text{(Thm statement)}$$

(b)

Apply $g_*(-)$

ok!

Take homology

$$\begin{tikzcd}
\Omega^\bullet_{\hat{X}}(*\hat{Y}) \arrow[r, "\sim"] \arrow[d] & \hat{f}_* \hat{\mathcal{Z}}^\bullet \arrow[d] \\
g_*(\Omega^\bullet_{\hat{X}}(*\hat{Y})) \arrow[r, "\therefore \sim"] \arrow[d, "\wr"] & g_*(\hat{f}_* \hat{\mathcal{Z}}^\bullet) \arrow[d, "\wr"] \\
\Omega^\bullet_X(*Y) \arrow[r, "\therefore \sim"] \arrow[d] & f_* \mathcal{Z}^\bullet \arrow[d] \\
\mathcal{H}^q(\Omega^\bullet_X(*Y)) \arrow[r, "\therefore \sim"] & R^q f_*(\underline{\mathbb{C}_U})
\end{tikzcd}$$

(b)

$\downarrow$

(a) Follows immediately, square of isos

$\downarrow$ + $\mathcal{E}$ argument

Then we win!

- **Claim:**

(b) induces an iso on cohomology sheaves, i.e. the theorem holds when

(X, Y, U) is replaced by $(\hat{X}, \hat{Y}, \hat{U})$

- Why? $\hat{X}$ is nonsingular, $\hat{Y}$ SNC, use an calculation by Atiyah-Hodge

- Locally $(\mathbb{D}^2 \setminus pt)^{\times k} \times (\mathbb{D}^2)^{\times \ell} \simeq \mathbb{T}^k$, look at
  - $\hookrightarrow$ generating cycles $\Delta_I := \gamma_{i_1} \times \gamma_{i_2} \times \cdots \times \gamma_{i_q}$, $1 \le i_j \le k \ \forall j$

    $\uparrow$ 1-cycle winding around z_{i_j}
  - $\hookrightarrow$ local meromorphic forms $\alpha = \alpha_0 + \sum_{k=1}^{r} \alpha_i / z_i^k$
- Compute generators for $\underbrace{H^q(X, \Omega_X(*Y))_x}_{\text{Analytic sheaf}} \hookrightarrow \underbrace{H^q(X; \tilde{\Omega}_X(*Y))}_{C^\infty \text{ sheaf}}$
- Shows isomorphism

germs of $\mathbb{C}$-valued C^∞ q-forms with arbitrary singularities along Y

- Thus

(a') $$\mathcal{H}^q(\Omega^\bullet_X(*Y)) \xrightarrow{\sim} R^q f_*(\underline{\mathbb{C}_U})$$

(b') $$\mathcal{H}^q(\Omega^\bullet_{\hat{X}}(*\hat{Y})) \xrightarrow{\sim} R^q \hat{f}_*(\underline{\mathbb{C}_{\hat{U}}})$$

$R^q(\text{LHS } *) = 0 = R^q(\text{RHS } *)$ (11)

(10') $\quad \Omega^\bullet_X(*\hat{Y}) \xrightarrow{\text{WTS}} f_*(\mathcal{L}^\bullet) \quad] \in \mathrm{Ch}(\mathrm{Sh}_{\hat{X}})$, $\mathbb{C} \hookrightarrow \mathcal{L}^\bullet$ an inj. resolution

$\Downarrow g_*(-)$ $\downarrow g_*(-)$ $\downarrow g_*(-)$

? $\quad g_*(\Omega^\bullet_X(*\hat{Y})) \longrightarrow g_*(f_*(\mathcal{L}^\bullet)) \quad] \in \mathrm{Ch}(\mathrm{Sh}_{\hat{X}})$

$\Downarrow$ iso $\downarrow \wr$ $\downarrow \wr$

(10) $\quad \Omega^\bullet_X(*Y) \longrightarrow f_*(\mathcal{L}^\bullet) \quad] \in \mathrm{Ch}(\mathrm{Sh}_X)$,

$\Downarrow$

(7) $\quad \underbrace{\mathcal{H}^q(\Omega^\bullet_X(*Y)) \xrightarrow{\therefore \text{iso}} R^q f_*(\underline{\mathbb{C}_U})}_{\text{Thm statement: this is an iso}} \quad] \in \mathrm{Ch}(\mathrm{Sh}_X)$

Fin!

Merci beaucoup à tous ☺