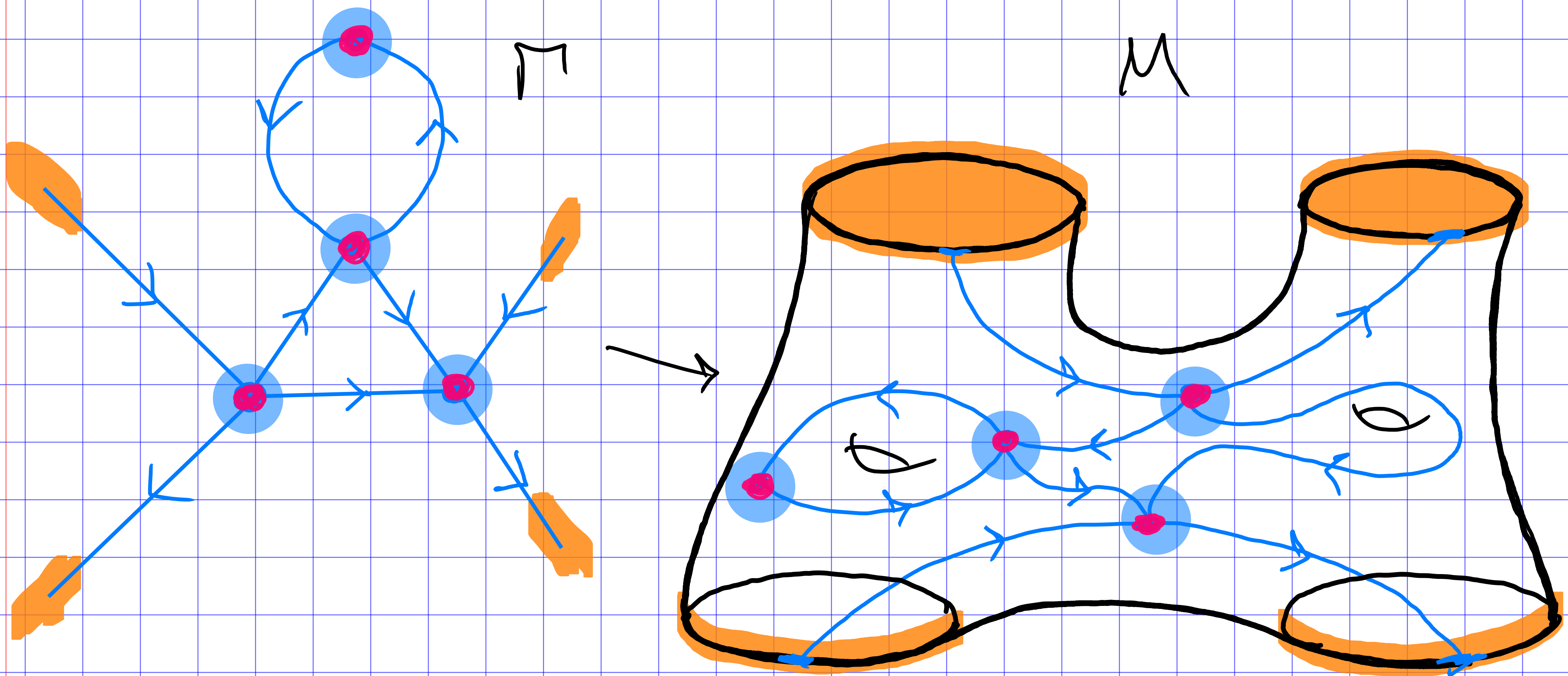
Graph Moduli Spaces & Cohomology Operations

(UGA, Fall 2021)



Motivation

- Study smooth 4-mfds using moduli spaces of anti-self-dual (ASD) instantons.

- Connections on $\underbrace{\begin{matrix}\mathcal{E}\\ \downarrow \\ X\end{matrix}}_{\in \mathrm{PrinBun}(G)/X}$: $\nabla : T_X \to \overbrace{\mathcal{D}^1(\mathcal{E}, \mathcal{E})}^{1^{st} \text{ order differential operators}}$

Contract for covariant deriv

or

$$\left\{ \begin{array}{l} \mathcal{E} \to \mathcal{E} \otimes_{\mathcal{O}_X} \Omega^1_X \\ s \mapsto \nabla(s) = a \otimes b \\ f.s \mapsto (df \otimes s) + f \underbrace{\nabla s}_{a \otimes b} \end{array} \right.$$

- Affine space of connections

$$\mathcal{A} \overset{\text{locally}}{\cong} \Omega^1(\mathcal{E}, \mathfrak{g}), \quad \mathfrak{g} = \mathrm{Lie}(G), \quad G = SO_3, SU_2, \text{ etc}$$

- Gauge transformations $\mathcal{G} := \left\{ \begin{array}{c} \mathcal{E} \xrightarrow{\psi} \mathcal{E} \\ \searrow \ \swarrow \\ X \end{array} \;\middle|\; \begin{array}{l} \psi \in \mathrm{Dif}(\mathcal{E}) \\ \forall x \exists y \\ \psi(\mathcal{E}_x) = \mathcal{E}_y \end{array} \right\}$.

Preserve fibers

- $\mathcal{A}/\mathcal{G}$: Moduli stack of connections on $\mathcal{E}$

- Curvature: $F_\nabla \in \Omega^2(\mathrm{ad}\,\mathcal{E})$, $\mathrm{ad}\,\mathcal{E} = \mathcal{E} \overset{G}{\times} \mathfrak{g}$

$$X, Y \mapsto (\nabla_X \nabla_Y - \nabla_Y \nabla_X) - \nabla_{[X,Y]}$$

↳ If $\nabla = A$ locally, $F_\nabla = dA + A \wedge A$

matrix of 1-forms — Apply componentwise

- Flatness: $F_\nabla = 0$

- Interpretations: ∇ = gauge potential, F_∇ = field strength

eg. components = elec/magnetic

Motivation

- $\star : \Omega^p X \to \Omega^{n-p} X$ where $n := \dim_{\mathbb{R}} X$

  Hodge star

- $n = 4 \Rightarrow \star : \Omega^2 X \circlearrowleft$ an operator

  Since $\star^2 = \mathrm{Id}_{\Omega^2 X}$, operator spectrum $= \{\pm 1\}$, so

$$\Omega^2 X = \Omega^{2,+} X \oplus \Omega^{2,-} X$$

+1 eigenspace, "self-dual"

−1 eigenspace "anti-self-dual"

- Induces $\mathcal{A} \cong \mathcal{A}^+ \sqcup \mathcal{A}^-$

$$\nabla \mapsto F^+_\nabla$$

$$\mathcal{A} \xrightarrow{s} \Omega^{2,+}(X, \operatorname{ad} \mathcal{E})$$

$s^{-1}(0)$: ASD solns

$$\Omega^2(X, \operatorname{ad} \mathcal{E}) \cong \Omega^{2,+}(X, \operatorname{ad} \mathcal{E}) \oplus \Omega^{2,-}(X, \operatorname{ad} \mathcal{E})$$

∇ is <u>anti-self-dual</u> $\iff \star F_\nabla = -F_\nabla \iff F_\nabla \in \mathcal{A}^-$

$\Rightarrow$ Study $\mathcal{M}^{ASD}(\mathcal{E}) := \mathcal{A}^- / \mathcal{G}$, moduli of $\underbrace{\text{ASD connections}}_{\text{instantons}}$

Motivation

- Key example: X^4 sm. Riemannian, $\pi_1 X = 1$, $\begin{matrix}\mathcal{E}\\ \downarrow \\ X\end{matrix} \in \mathrm{Prin\,Bun}(SU_2)$

$$\mathcal{M}(\mathcal{E}, c_2) := \{ F_\nabla \in \mathcal{M}^{ASD}(\mathcal{E}) \mid c_2(\mathcal{E}) = c_2 \}$$

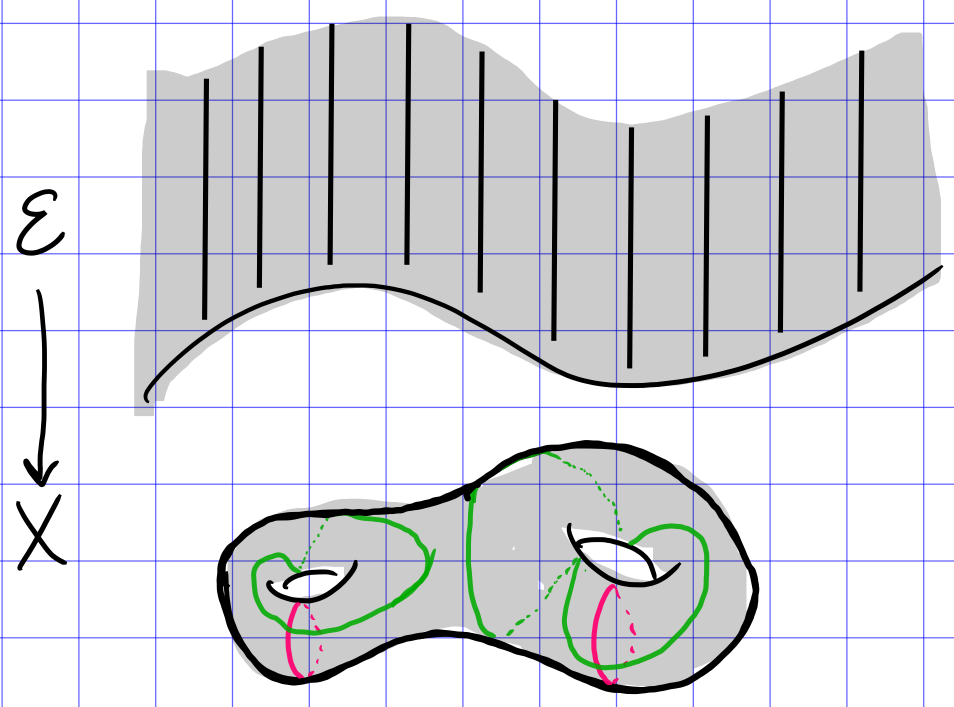
(Note c_2 classifies principal SU_2 bundles)

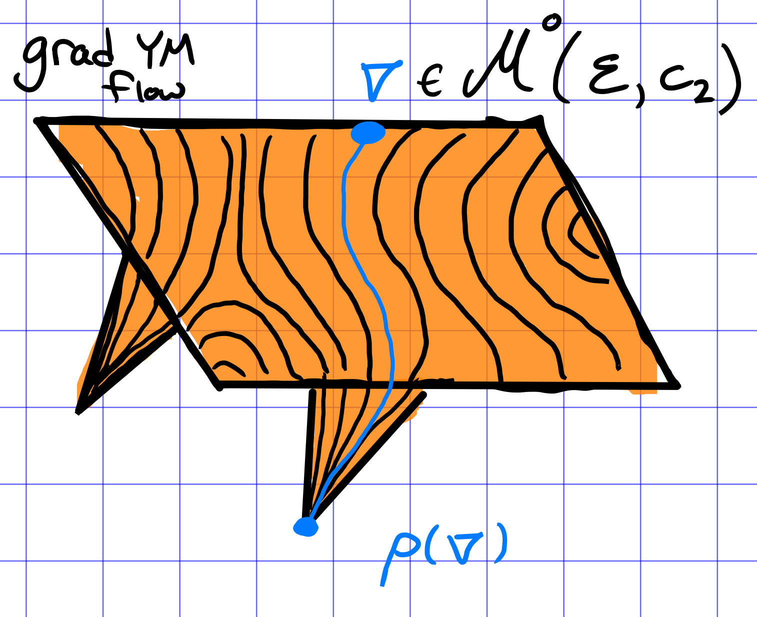
- Minimize an action functional

induced by the metric g

$$YM : \mathcal{M}(\mathcal{E}, c_2) \longrightarrow \mathbb{R}$$
$$\nabla \longmapsto \int_X \|F_\nabla\|^2 \, dV = \int_X \mathrm{Tr}(F \wedge \star F) \, dV$$

- Take $\mathcal{E} = X \times SU_2$ trivial, gradient flow of YM ψ_t yields $\nabla \xrightarrow{t \to \infty} \rho(\nabla) \in CF_\bullet(X)$ a flat connection





minimizes action

- Donaldson invariant:

$$q_0(X) = \sum_{\nabla \in \mathcal{M}^0(X, \mathcal{E})} 1 \cdot [\rho(\nabla)] \in HF_\bullet(\partial X)$$

$$\in \Big(\bigotimes_{\text{inward}} HF^\bullet(X_i)\Big) \otimes \Big(\bigotimes_{\text{outward}} HF_\bullet(X_i)\Big) \text{ when } \partial X = \bigsqcup X_i$$

- Then use $V^\vee \otimes W \cong \mathrm{Hom}_{k\text{-Mod}}(V, W)$

to realize as cohomology operations

$$HF^\bullet(X) \xrightarrow{q(X)} HF^\bullet(X).$$

Today

Q: What is the homotopy type of $\mathcal{M}$?

Q: What is the homotopy type of the "ends" of $\mathcal{M}$?

Q: Are there higher Donaldson invariants?

Today: Cohomology operations from equivariant theories

- The simplest case when $HF_\bullet(\mathcal{M}^{ends}) \cong H^\bullet_{sing}(\mathcal{M}^{ends})$.
- Show Donaldson invariants on $\mathcal{M}$ recover standard cohomology operations
  (Deform the M-structures)

Equivariant Cohomology

- **Def:** For $G \in \text{AlgGrp}$, $X \in G\text{-Top}$, the G-equivariant homology is

$$H^*_G(X;\mathbb{Z}) := H^*_{sing}(EG \times^G X; \mathbb{Z})$$

$$\Big\downarrow \text{ }\cong \quad \text{if } G \curvearrowright X \text{ freely}$$

$$H^*_{sing}(X/G;\mathbb{Z})$$

$$\left(EG \times^G X = X/\!\!/G, \ \underline{\text{the homotopy quotient}} \right)$$

- Notation: $C_p = \mathbb{Z}/p\mathbb{Z}$

Cohomology Operations

- A **natural transformation**: $\eta : E(-) \longrightarrow E'(-)$

  or if E, E' are representable, a morphism

$$B\eta \in [BE, BE']_{hoTop}$$

- **Eg** **cup^{sq}**: $H^n(-) \to H^{2n}(-)$

$$\alpha \mapsto \alpha^2 := \alpha \cup \alpha$$

Unstable

$$\begin{tikzcd} H^n(X) \arrow[r, "\mathrm{cup}^{sq}"] \arrow[d, "\cong"] \arrow[dr, phantom, "\times \text{ not commutative}"] & H^{2n}(X) \arrow[d, "\cong"] \\ H^{n+1}(\Sigma X) \arrow[r, "\mathrm{cup}^{sq}"'] & H^{2n+1}(\Sigma X) \end{tikzcd}$$

Stable $\downarrow$

- **Eg** **Sq^i**: $H^n(-; C_p) \to H^{n+i}(-; C_p)$

  Sq^1: Bockstein:

$$0 \to C_p \to C_{p^2} \to C_p \to 0$$

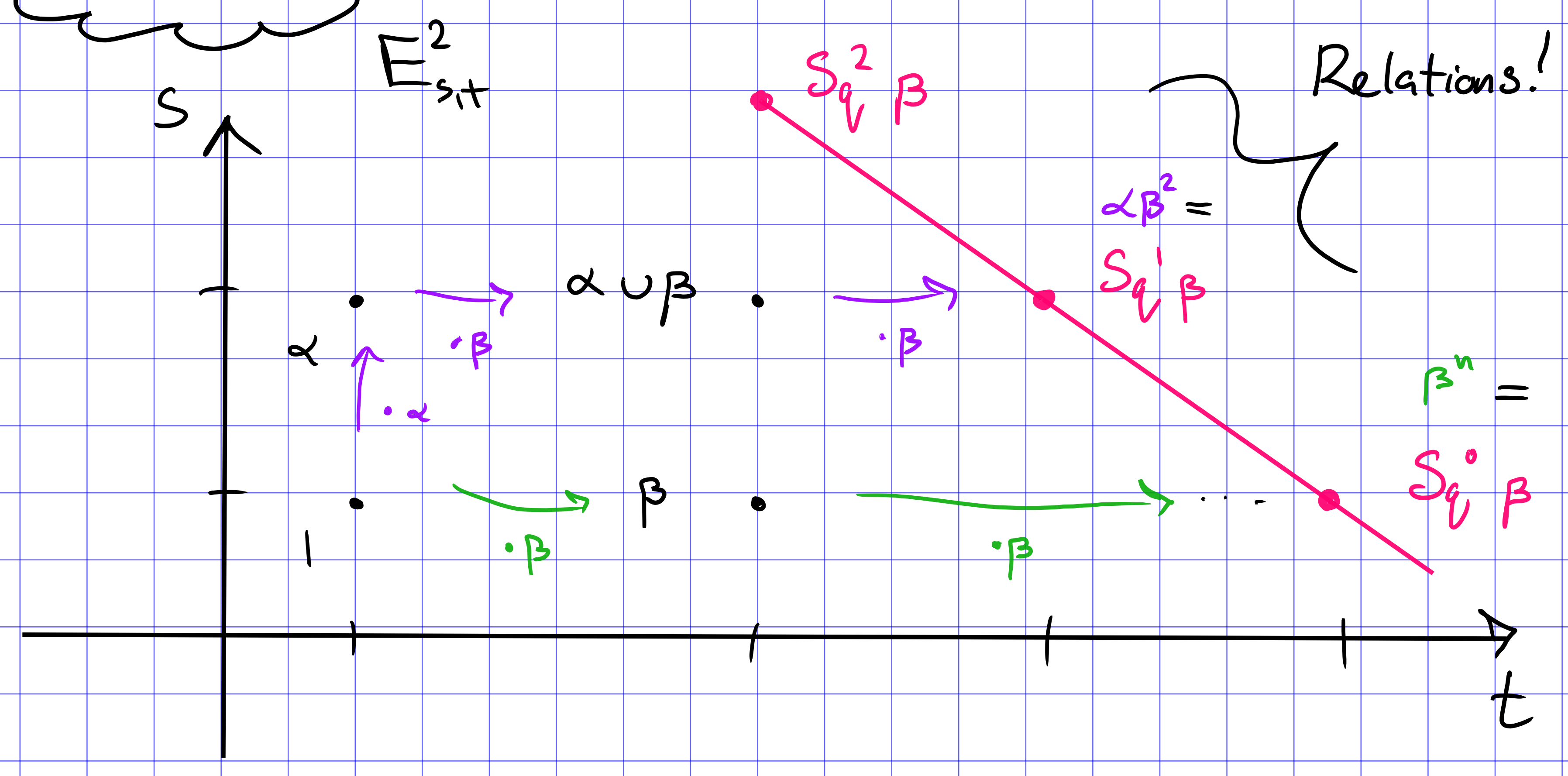
$$\Rightarrow H^i(X; C_p) \to H^i(X; C_{p^2}) \to H^i(X; C_p) \xrightarrow{\beta} H^{i+1}(X; C_p)$$

- **Cartan**: $Sq^n(x \cup y) = \sum_{i+j=n} Sq^i(x) \smile Sq^j(y)$

- **Adem**: $Sq^i Sq^j = \sum_{k=0}^{i/2} \binom{j+k-1}{i-2k} Sq^{i+j-k} Sq^k$

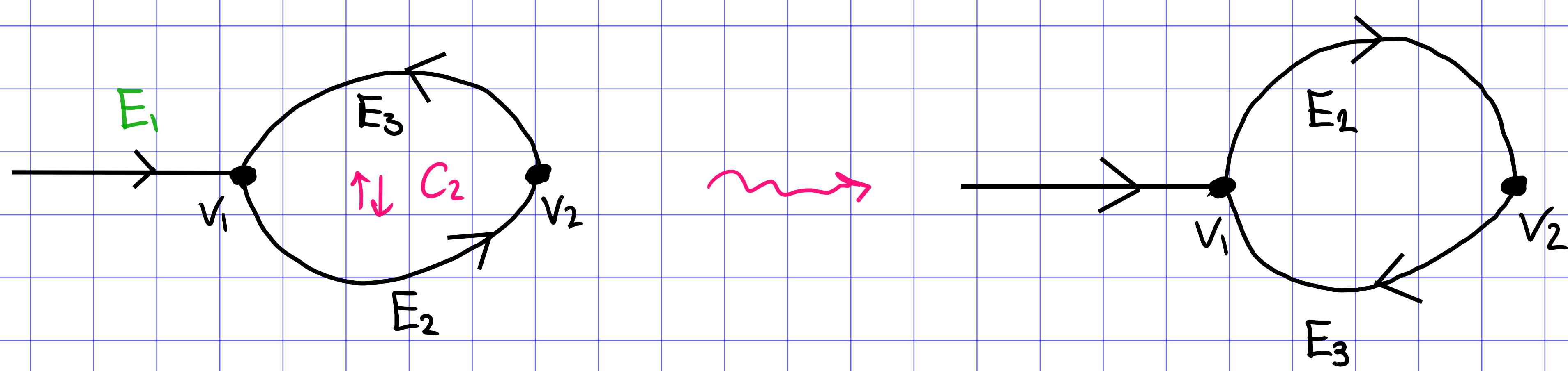
- **Why care**: Spectral sequence computations!

A cartoon:



Examples

Ex (Stiefel-Whitney)



· Note $C_2 \curvearrowright E_1$ trivially $\Rightarrow Q(\Gamma, M) \in H^{\bullet}_{C_2}(BC_2 \times M)$

So for each $\alpha_i \in H_i(BC_2; C_2) \cong C_2$

$BC_2 \simeq \mathbb{RP}^\infty$

$$\Big\downarrow \text{Take invariant } q$$

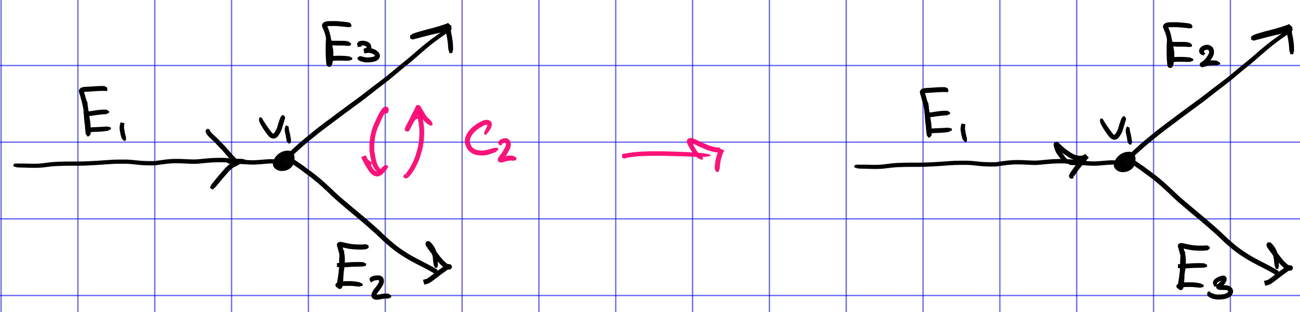
$$q_{\alpha_i}(\Gamma, M) \in H^{d-i}(M; C_2)$$

Claim: $[q_{\alpha_i}(\Gamma, M)] = \underbrace{[w_{d-i}(TM)]}$

The d-i th Stiefel-Whitney class.

Examples

Ex Cup products



For each $\alpha_i \in H_i(BC_2; C_2) \cong C_2$

Take invariant q

$$q(\alpha_i) \in \bigoplus_{\ell} \left[H^{C_2}_{\ell}(M^{\times 2}; C_2) \otimes H^{\ell - i}(M; C_2) \right]$$

$$\in \left[H^{\bullet}_{C_2}(M^{\times 2}; C_2) \longrightarrow H^{\bullet - i}(M; C_2) \right]$$

$\Rightarrow q(\alpha_i) \cong \mathrm{cup}_{(-i)}$

- Note

$$Sq^{k-i} = \left[H^{\bullet}_{C_2}(M^{\times 2}; C_2) \xrightarrow{q} H^{\bullet - i}(M; C_2) \xrightarrow{\mathrm{cup}^{sq}} H^{2(\bullet - i)}_{C_2}(M^{\times 2}; C_2) \right]$$

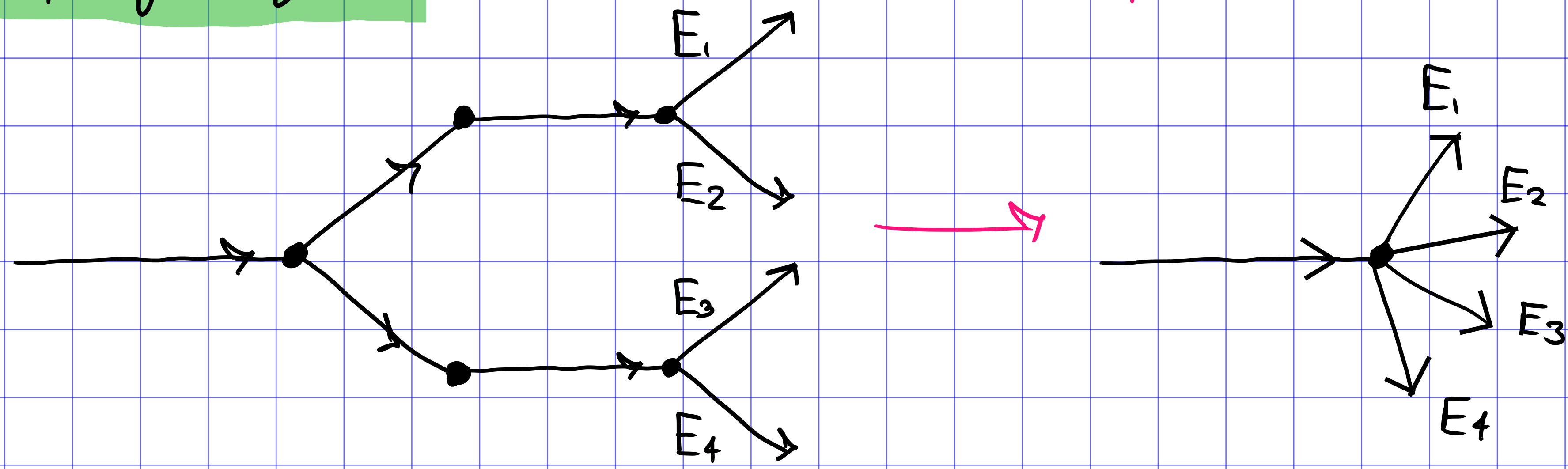
$$\alpha_i \mapsto q(\alpha_i), \qquad \beta \mapsto \beta^{\otimes 2}$$

So $q(\alpha_i) \circ \mathrm{cup}^{sq} \cong Sq^{k-i} \circ \mathrm{cup}^{sq} \Rightarrow q(\alpha_i) \cong Sq^{i}$!

Examples

Composing Sq^i:

Homotopy equivalence of moduli spaces



- Claim: D_4 symmetry ? Has S_4 sym. ✓

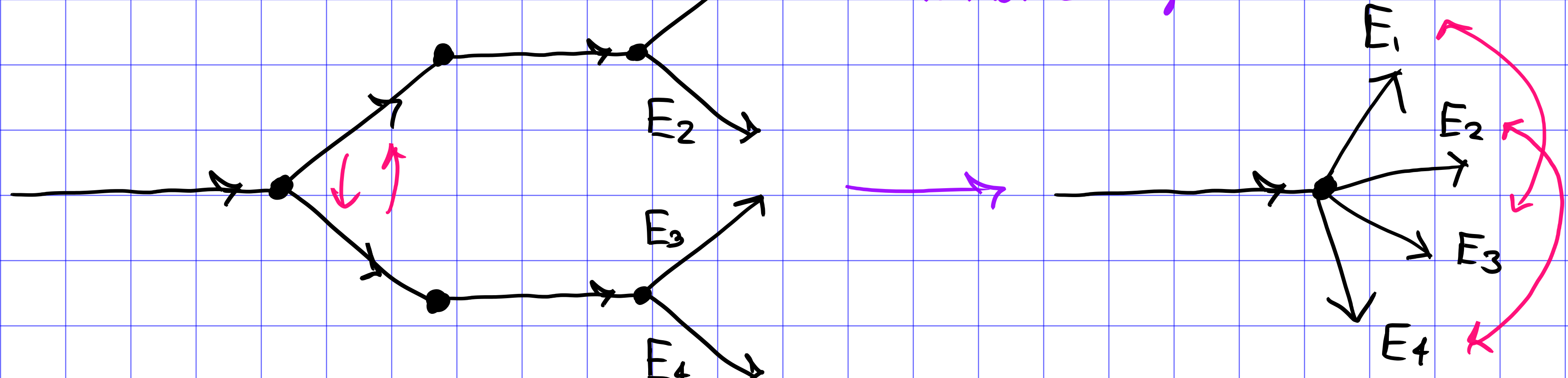
- q_{D_4} factors through q_{S_4} $\Rightarrow$ Adem relations

$$Sq^i Sq^j = \sum_{k=0}^{i/2} \binom{j+k-1}{i-2k} Sq^{i+j-k} Sq^k$$

- Composing with cup product yields Cartan relations

$$Sq^n(x \smile y) = \sum_{i+j=n} Sq^i(x) \smile Sq^j(y)$$

Contract 2 edges



C_2 symmetry $\xrightarrow{\text{diagonal}}$ $C_2 \times C_2$ symmetry

- The kernel of the associated homology map yields the Cartan relations.

Setup

Setup: Γ as before, $G \leq \underset{\text{Graph}}{\mathrm{Aut}\,\Gamma}$.

- Then

$$G \curvearrowright \left\{ \begin{array}{c} \mathcal{M}(\Gamma, M) \\ \pi \downarrow \\ S(\Gamma, M) \end{array} \right. \quad \xrightarrow{G\text{-equiv}} \quad \begin{array}{c} \mathcal{M}(\Gamma, M)/G \\ \pi \downarrow \\ S(\Gamma, M)/G \end{array}$$

- G action free since f_i are distinct, $\mathcal{M}(\Gamma, M)/G \simeq pt$

$$\Rightarrow \mathcal{M}(\Gamma, M) \simeq EG \Rightarrow S(\Gamma, M) \simeq EG/G \simeq BG$$

- Replace $S(\Gamma, M)$ by hty equiv space $S_f(\Gamma, M) \simeq BG$ where

$$S_f(\Gamma, M) = \{ \sigma \in S(\Gamma, M) \mid f_i \in \mathcal{U}_f \} \quad \text{for a fixed } f.$$

- Get

$$\begin{array}{ccc} \mathcal{M}_f(\Gamma, M) := \pi^{-1}(S_f(\Gamma, M)) & \xrightarrow{\subseteq} & \mathcal{M}(\Gamma, M) \\ \pi \downarrow & & \downarrow \pi \\ S_f(\Gamma, M) & \xrightarrow{\subseteq} & S(\Gamma, M) \end{array}$$

Where

- $f \in \mathcal{U}_f \subseteq C^\infty(M, \mathbb{R})$ a contractible nbhd of Morse-Smale fns.
- $g \in \mathcal{U}_f \Rightarrow \mathrm{Crit}(g)$ isolated. $\mathcal{U}_f \simeq pt \Rightarrow \mathrm{crit}(g) \underset{\text{Set}}{\cong} \mathrm{crit}(f)$

Flow along contraction

Simplicial Reduction

- How to distinguish elts in $\mathcal{M}_f(\Gamma, M)$ by asymptotics @ ends:

  $G \to \mathrm{Aut}_{set}(\{a_i, \text{ crit pts to which } \Gamma\text{-flows converge}\})$

For $\gamma \in \mathcal{M}_f(\Gamma, M)$, let $\vec{a}$:= orbit of crit pts

$\uparrow$ "critical orbit"

$M_f(\Gamma, M, \vec{a}) \subseteq \mathcal{M}_f(\Gamma, M)/G$ whose crit orbits correspond to $\vec{a}$.

- Take pullback

$$\begin{tikzcd}
\mathcal{M}^i_\delta(\Gamma, M, \vec{a}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{M}_f(\Gamma, M)/G \arrow[d, "\pi"] \\
\Delta^i \arrow[r, "\delta"'] & S_f(\Gamma, M)/G
\end{tikzcd}$$

Thm

$\delta \in C^0(\Delta^i, S_f/G)$ generic $\Rightarrow \mathcal{M}^i_\delta \in$ sm Mfd. oriented.

$\dim_{\mathbb{R}} M^i_\delta = \dim \mathcal{M}_f + i$

(Typo....)

Thm

$M^{i,0}_\delta$ (the zero dim spaces) are cpt $\Rightarrow \# \mathcal{M}^{i,0}_\delta < \infty$.

Constructing Q

$\underbrace{(M\times M\times\cdots\times M)}_{n_1 \text{ copies}}\times\underbrace{(M\times M\times\cdots)}_{n_2 \text{ copies}}$

- Constructing $\underbrace{Q = Q(\Gamma, M)}_{\text{intermediate step}} \in H^*(EG \overset{G}{\times} \underbrace{(M^{\times n_1}\times M^{\times n_2})}))$

  A G-space via

  $Aut(\Gamma) \to Aut(\Gamma_{n_1})\times Aut(\Gamma_{n_2})$

  Symmetry $\mapsto$ perms of ends

- For $p \in S_f(\Gamma, M)/G$ generic $\boxed{Q_p(\Gamma, M)} \in \operatorname{Hom}_{G\text{-Mod}}\left(C_\bullet(\tilde{p}),\ C^\bullet(M,f)^{\otimes n_1}\otimes C_\bullet(M,f)^{\otimes n_2}\right)$

  where $\tilde{p}\in S_f(\Gamma, M)$ is a lift.

- How to define the class $\boxed{Q_p(\Gamma, M)}$

  . $\forall$ simplices $\Delta^i \xrightarrow{\partial} p$, consider $\{\vec{a}\in crit(f) \mid \dim_{\mathbb{R}} \mathcal{M}_S(\Gamma, M, \vec{a}) = 0\}$.

$$\begin{array}{ccc} \delta & \xrightarrow[G\text{-equiv.}]{\boxed{Q_p(\Gamma,M)}} & \sum \# \mathcal{M}^0_S(\Gamma, M, \vec{a})\cdot[\vec{a}] \\ \cap & & \cap \\ C_\bullet(\tilde{p}) & \longrightarrow & C^\bullet(M,f)^{\otimes n_1}\otimes C_\bullet(M,f)^{\otimes n_2} \end{array}$$

Lem: $\boxed{Q_p(\Gamma, M)}$ is a G-equiv cocycle

$$\begin{array}{l} \Rightarrow \in H^\bullet(\tilde{p} \overset{G}{\times} (M^{\times n_1}\times M^{\times n_2})) \\ \quad\downarrow \\ H^\bullet(S_f(\Gamma, M) \overset{G}{\times} (M^{\times n_1}\times M^{\times n_2})) \\ \quad \wr\wr ? \\ H^\bullet(EG \overset{G}{\times} (M^{\times n_1}\times M^{\times n_2})) \end{array}$$

Valid $\forall$ finite subcomplexes
doesn't depend on perturbations

?

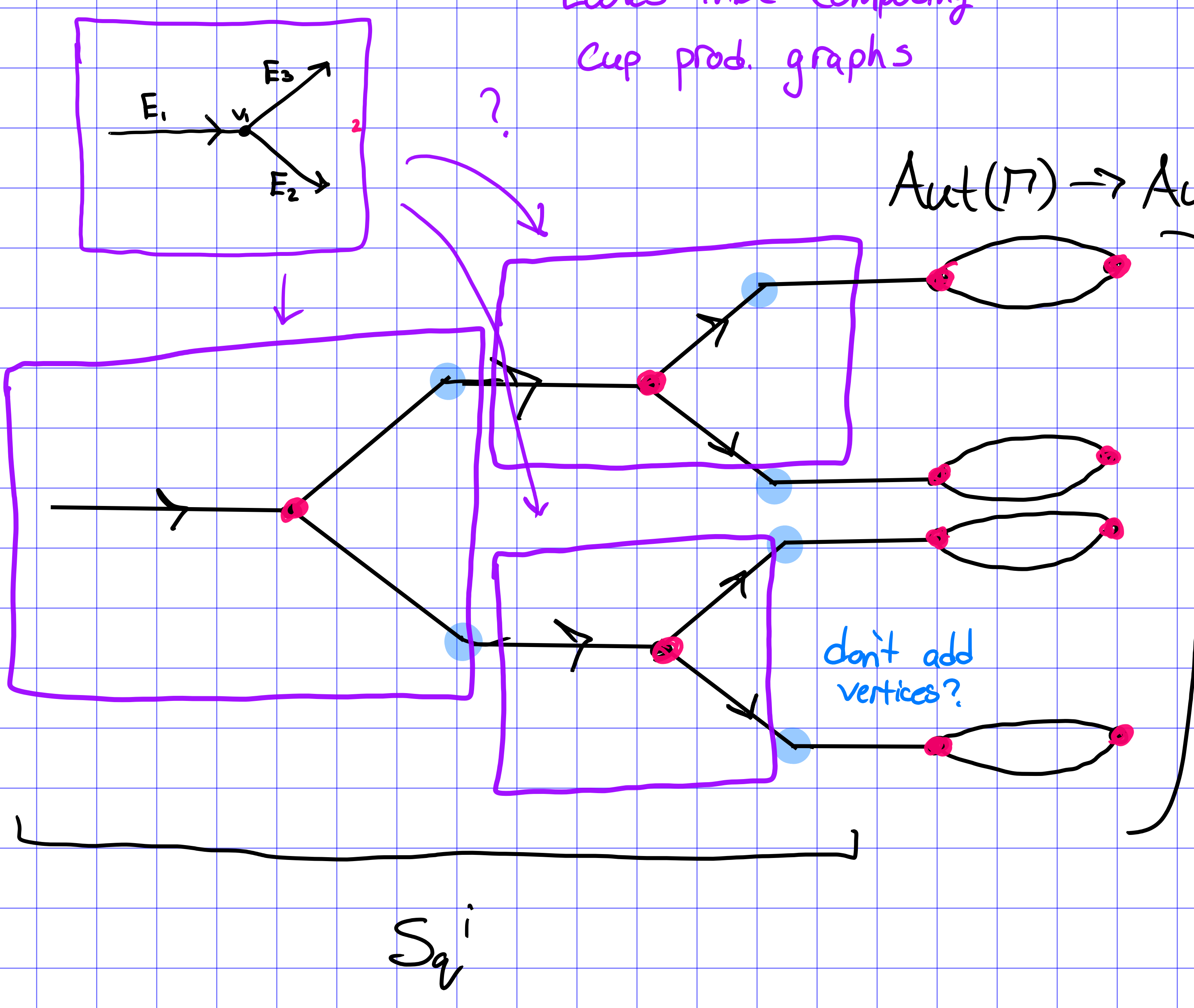
Def (5)
For $\alpha \in H_\bullet(BG)$ define $\boxed{q(\Gamma, M)(\alpha)} \in H^\bullet_{G_{n_1}}(M^{n_1}) \otimes H_\bullet^{G_{n_2}}(M^{n_2})$

by "eval $Q(\Gamma, M)$ on α, apply $G \to G_{n_1}\times G_{n_2}$".

Q: How to compose graphs to compute operations ...?

Looks like composing cup prod. graphs

$Aut(\Gamma) \to Aut(\Gamma_{n_1})\times Aut(\Gamma_{n_2})$



Closed end is like "evaluation"?