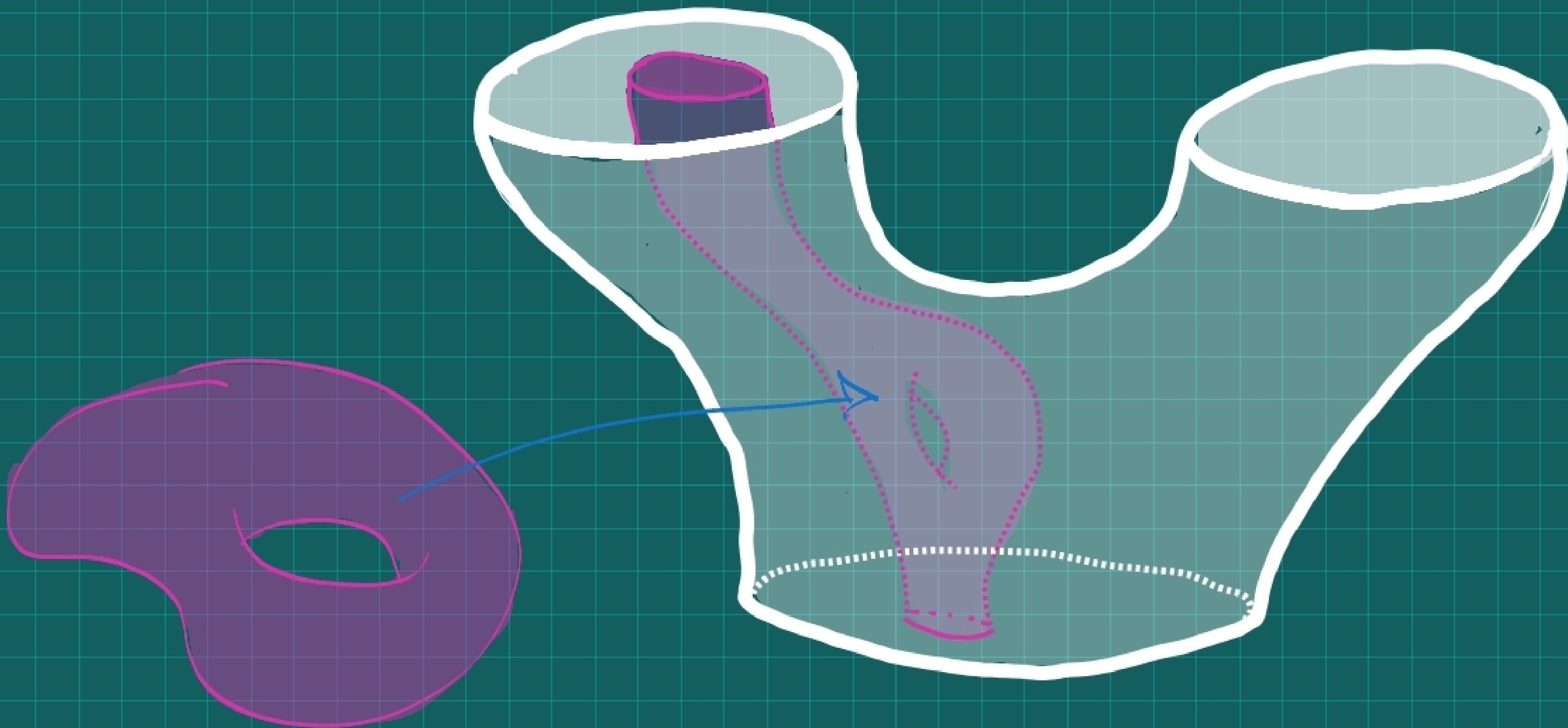
Reading Seminar

J-Holomorphic Curves



Overview & Goals

Eg: $Gr_k(\mathbb{C}^\infty)$ classifies $Bun_{\mathbb{C}}$

- Long-term goal: define the Gromov-Witten invariants $GW(M, A, g, n)$
  - ↳ For (M, ω) compact symplectic, $[A] \in H_2^{sing}(M; \mathbb{Z})$, define a moduli space

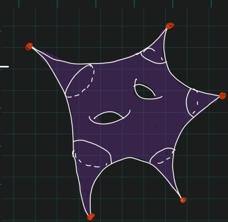
$$\overline{\mathcal{M}}_{g,n}(A, J) := \left\{ \begin{array}{l} J\text{-holomorphic curves } \Sigma \xrightarrow{f} M \text{ with } [\Sigma] = [A] \in H_2 \\ \text{where } g(\Sigma) = g \text{ and } [x_1, \cdots, x_n] \in \Sigma^{\times n} \text{ are } n \text{ marked pts} \end{array} \right\} / \sim$$

ev_j : $f(x_i) \in M$

forget A, J & J-Holo condition

$$\overline{\mathcal{M}}_{g,n} = \{ \text{Stable curves } \Sigma \hookrightarrow M \text{ with } g(\Sigma) = g \text{ \& } n \text{ marked pts} \}$$

A smooth Deligne-Mumford stack $\rightsquigarrow$ a $\mathbb{C}$-orbifold. For $g = 0$: a compact smooth $\mathbb{C}$-mfd!



Fixed points of a group action

- Rough def:

$$GW(M, A, g, n) : H^*(M; \mathbb{Q})^{\otimes n} \otimes_{\mathbb{Z}} H^*(\overline{\mathcal{M}}_{g,n}; \mathbb{Q}) \longrightarrow \mathbb{Q}$$

$$[\alpha_1, \alpha_2, \cdots, \alpha_n] \otimes \beta \longmapsto \int_{\overline{\mathcal{M}}_{g,n}(A,J)} \prod_{j=1}^{n} ev_j^*(\alpha_j) \smile \pi^*(\beta^{PD})$$

Idea: Push cohom classes into $\overline{\mathcal{M}}_{g,n}$ and "cap with fundamental class" (integrate over moduli space)

$$= \# \left\{ \begin{array}{l} J\text{-holomorphic curves } \Sigma_g \\ \text{with } n \text{ marked pts } x_i \\ \text{where } x_i \text{ intersects a cycle} \\ \quad X_i \subseteq H_2(M) \text{ dual to } \alpha_i \\ \text{and } [\pi \Sigma_g] = [\beta] \in H_2(\overline{\mathcal{M}}_{g,n}) \end{array} \right\}$$

- Example: $\# \left\{ \begin{array}{l} \text{Lines in } \mathbb{P}^3 \text{ intersecting} \\ 4 \text{ (generic) lines} \end{array} \right\} = GW^{\mathbb{P}^3(\mathbb{C})}_{L,4}(c^2, c^2, c^2, c^2) = 2$

- These will only depend on the deformation type of (M, ω)

Can take 1-param. families $\gamma : I \to \Omega^2(M)$, $t \mapsto \omega_t$ $\Big\}$ (M, ω_t) of "semipositive" sympl. mfds

- In dim $M = 4$, depend only on the diffeomorphism type and are iso to SW invariants.
  (Counterexamples in dim $M = 6$)

- Remarks
  - $g = 0$: Quantum cohomology & Frobenius manifolds
  - Can produce a TQFT (somehow)
  - Mirror symmetry: conjecturally gives a 2^{nd} way to compute GW.

Ch. 3: Moduli Spaces & Transversality

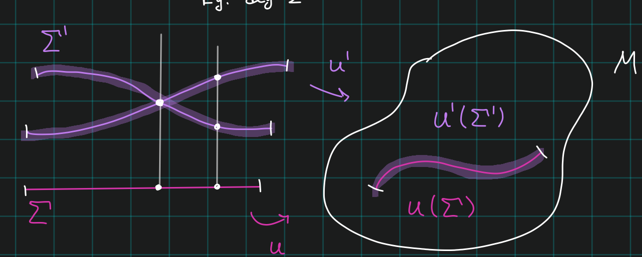
- Ch. 3 Goal: Show that for generic J, $\mathcal{M}^*(A,\Sigma,J) \in \text{sm}\,\text{Mfd}_{\mathbb{C}}^{<\infty}$ (for simple curves, easier transversality)
- 3.1: Moduli space of simple curves
- 3.2: Thom-Smale transversality
  ↳ $\Rightarrow \mathcal{M}^*(A,\Sigma,J)$ is a smooth mfd when the linearized operator is surjective $\forall$ J-hol. curves
- 3.3: Examples in dim 4, surjectivity $\forall J$ (use RR instead of Sard-Smale)
- 3.4: Moduli spaces with pointwise constraints

- Recall from Han's talks:
  - $(M,\omega,J) \in \text{Mfd}^{2n}(\text{Sp})$, J an ω-tame almost $\mathbb{C}$-struct on M
  - (Σ, j_Σ, dV) a Riemann sfc, compact
  - The CR operator $\bar\partial_J = \frac{1}{2}(Jd - dj_\Sigma)$, $\operatorname{Ker}\bar\partial_J$ = J-Holomorphic curves.
  - $u \in C^\infty(\Sigma, M)$ a curve/solution
  - $\mathcal{M}^*(A,\Sigma,J) := \left\{ u \in C^\infty(\Sigma,M) \cap \operatorname{Ker}\bar\partial_J \;\middle|\; \begin{array}{l}[u(\Sigma)] = A \in H_2(M;\mathbb{Z}) \\ \& \ u \text{ simple}\end{array} \right\}$

  where u is <u>not simple</u> iff $\exists$
$$\begin{tikzcd} \tilde\Sigma \arrow[d, "\text{branched deg} \geq 2"'] \arrow[dr, "u'"] & \\ \Sigma \arrow[r, "u"'] & M \end{tikzcd}$$

  (factors through a ramified curve)

Eg. deg 2



$\approx$ orbifold points in $\mathcal{M}(A,\Sigma,J)$

- $$\begin{tikzcd} \mathcal{E}_u := \Omega^{0,1}(\Sigma, u^*TM) \arrow[r] & \mathcal{E} \arrow[d] \\ & \mathcal{B} := \{u \in C^\infty(\Sigma, M) \mid [u(\Sigma)] = A\} \arrow[u, bend right, "s(u) := (u,\, \bar\partial_J(u))"'] \end{tikzcd}$$

  $\mathcal{M}(A,\Sigma,J) = \mathcal{Z}(s)$

  obtain $\mathcal{M}^*$ by taking $\mathcal{B} \cap \{u \mid \text{somewhere inj.}\}$

- $$\begin{tikzcd} \mathcal{E} \arrow[d] \\ \mathcal{B} \end{tikzcd} \overset{T}{\rightsquigarrow} \begin{tikzcd} T_{(u,0)}\mathcal{E} \arrow[d] & = T_u\mathcal{B} \oplus \mathcal{E}_u \arrow[dl, "\pi_{\mathcal{B}}"'] \arrow[dr, "\pi_u"] & \\ T_u\mathcal{B} \arrow[u, bend left, "ds"] \arrow[rr, bend right, ":= D_u"'] & T_u\mathcal{B} & \mathcal{E}_u \end{tikzcd}$$

  $:= D_u : T_u\mathcal{B} \to \mathcal{E}_u$

  $= D_u : \Omega^0(\Sigma; u^*TM) \to \Omega^{0,1}(\Sigma; u^*TM)$

  (Complete to Sobolev spaces $W^{k,p}$)

  The <u>vertical differential</u>

  Transversality $\Leftrightarrow$ D_u is surjective

- D_u is <u>Fredholm</u>: $\dim_{\mathbb{R}} \operatorname{Ker} D_u$ & $\dim_{\mathbb{R}} \operatorname{Coker} D_u < \infty$, im D_u closed,

  $\operatorname{Ind}(D_u) := \dim\operatorname{Ker} D_u - \dim\operatorname{coker} D_u < \infty$ is well-defined.

  $\overset{RR}{=} n(2 - 2g(\Sigma)) + 2c_1(u^*TM)$. $\left(= \chi(H^*_{\bar\partial}\, u^*TM),\ \Gamma(\Omega^{p,q}) \xrightarrow{\bar\partial} \Gamma(\Omega^{p,q+1}) \right)$

- Upshot: For J a regular value of

$$\begin{tikzcd} \mathcal{M}^*(A,\Sigma,\mathcal{J}^\ell) = \{(u,J) \in \mathcal{M}^* \times \mathcal{J}^\ell\} \arrow[d, "\pi"] \\ \mathcal{J}^\ell \end{tikzcd}$$

  ↖ universal moduli space

  $\mathcal{J}^\ell = \{J \in C^\ell(TM, TM)\}$ with the C^∞ topology

  $\pi^{-1}(J) = \mathcal{M}^*(A,\Sigma,J)$ is a fin dim manifold (implicit fn thm for Banach mfds)
  and $T_{(u,0)}\mathcal{M}^*(A,\Sigma,J) = \operatorname{Ker} D_u$

  Sard-Smale: Regular values are (Baire) 2nd category in $\mathcal{J}^\ell$

3.3: Regularity

$D_u : T_u\mathcal{B} \to \mathcal{E}_u$
$= D_u : \Omega^0(\Sigma; u^*TM) \to \Omega^{0,1}(\Sigma; u^*TM)$

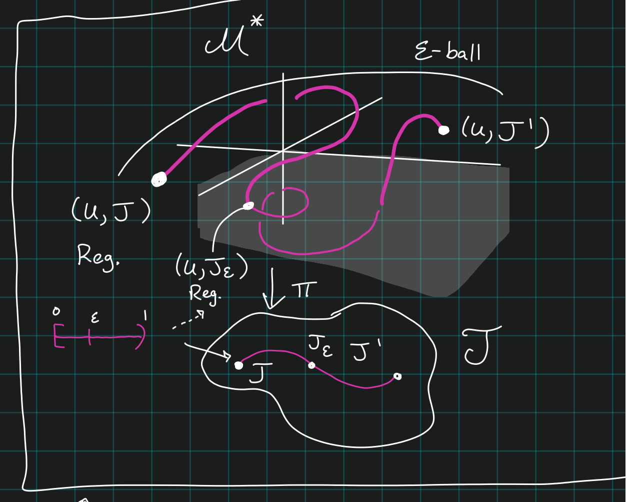
- Def (3.1.4): J is regular for A $\iff$ D_u is surjective $\forall u \in \mathcal{M}^*(A, \Sigma, J)$

- A point $p := (u, J) \in \mathcal{M}^*(A, \Sigma, \mathcal{J})$ is regular $\iff$ $T_p\mathcal{M}^*(A,\Sigma,\mathcal{J}) \xrightarrow{d\pi_p} T_J\mathcal{J}$ surjects

$\Rightarrow$

$$\begin{array}{ccc} & & \mathcal{M}^*(A,\Sigma,\mathcal{J}) \\ & {\scriptstyle t \mapsto (u_t, J_t)} \nearrow \exists\tilde{\gamma} & \downarrow \pi \\ [0,\varepsilon) & \xrightarrow{\gamma} & \mathcal{J} = \left\{ \begin{array}{c}\text{All almost-}\mathbb{C} \\ \text{structures } J\end{array} \right\} \\ & t \mapsto J_t & \\ & 0 \mapsto J & \end{array}$$

Smooth paths can be lifted
So if (u,J) is regular
then $\exists\, (u, J')$ regular nearby



⚠ (u,J) not regular may imply no nearby regular pts!

- Main Thm (3.3.4): Let $(\overbrace{\Sigma}^{\text{sphere}}, j_\Sigma) \hookrightarrow (M^4, J)$ be a holo. embedded sphere with $\overbrace{\Sigma^2 := \Sigma\cdot\Sigma}^{\text{self-intersection}} := p$
Then J is regular for $A := [\Sigma]$ $\iff$ $p \geq -1$.

- Main Thm (3.3.5): Let $\tilde{A} := [S^2 \times pt] \in H_2(\overbrace{S^2 \times M}^{\tilde{M}}; \mathbb{Z})$. $M \in \mathrm{Mfd}(S_p)$
Then $\forall J \in \mathcal{J}(M,\omega)$, $\tilde{J} := i \times J$ is regular for $\tilde{A}$

- Summary of other lemmas: criteria for surjectivity

  - 3.3.1: $c_1(L_k) \geq -1$ when...
    $u^*TM = \oplus L_k$
    for $\mathbb{P}^1_{\mathbb{C}} \xrightarrow{u} M$ & J integrable

  - 3.3.2: $c_1(L_k) \geq -1$ when...

    $\mathrm{Bun}_{\mathbb{C}} \left[\begin{array}{c} \mathcal{E} \\ \downarrow \\ \mathbb{P}^1_{\mathbb{C}} \end{array} \right.$ with $\mathcal{E} \overset{\exists}{=} \oplus L_k$, $D : \Omega^0(\mathbb{P}^1; \mathcal{E}) \to \Omega^{0,1}(\mathbb{P}^1; \mathcal{E})$ an $\mathbb{R}$-linear CR operator,
    $D(L_k) \subseteq L_k \ \forall k$

  - 3.3.3: $c_1(u^*TM) \geq 1$ when...
    (M^4, J) almost complex, $u \in \mathrm{Imm}^\infty(\mathbb{P}^1_{\mathbb{C}}, M)^{J\text{-Hol}}$

- Rest of the chapter: examples of non-regular curves.

3.3: Regularity

$$\begin{tikzcd} u^*TM \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & TM \arrow[d] \\ \Sigma \arrow[r] & M \end{tikzcd}$$

- Fix $\Sigma := \mathbb{P}^1_{\mathbb{C}} \cong S^2$, write $c_1(L) := \langle c_1(L), [\Sigma] \rangle$, assume J is integrable
- Thm (Grothendieck): in $\mathrm{Bun}^r_{\mathbb{C}}(\mathrm{Holo})$,

$$\begin{tikzcd} \mathbb{C}^r \arrow[r] & \mathcal{E} \arrow[d] \arrow[r, squiggly, "\text{splitting principle}"'] & \mathbb{C}^r \arrow[r] & \bigoplus_{k=1}^{r} \mathcal{O}_{\mathbb{P}^1}(n_k) \arrow[d] \\ & \mathbb{P}^1 & & \mathbb{P}^1 \end{tikzcd}$$

unique

$$\boxed{\Rightarrow \ u^*TM = \bigoplus_{k=1}^{\ell} L_k .}$$

- Lemma (3.3.1): If $c_1(L_k) \geq -1 \ \forall k$ then D_u is surjective.
- Proof (1: AG)
  - Thm (RR, C.1.10 part 3): If $\begin{matrix}\mathcal{E} \\ \downarrow \\ \Sigma\end{matrix} \in \mathrm{Bun}_{\mathbb{C}}(\mathrm{Holo})$ and $\mathcal{F} \subseteq \mathcal{E}$ is a sub-bundle,

$$\boxed{D_u \text{ is surjective} \iff \overbrace{\mu(\mathcal{E}, \mathcal{F})}^{\text{Maslov index}} + 2\chi(\Sigma) > 0}$$

Moreover $\boxed{\mu(\mathcal{E}, \emptyset) = 2\langle c_1(\mathcal{E}), [\Sigma] \rangle}$ when $\partial\Sigma = \emptyset$.

  - Now expand:

$\mu(u^*TM, \emptyset) + 2\chi(\Sigma)$ $\longrightarrow$ Σ a sphere $\Rightarrow H^*(S^2; \mathbb{Z}) = \mathbb{Z}[x]/\langle x^2 \rangle$, $|x| = 2$

$\Rightarrow \chi(\Sigma) = 1 - 0 + 1 = 2$

$= \mu\left(\bigoplus_{k=1}^{2} L_k, \emptyset\right) + 4$

$= \sum_{k=1}^{2} \mu(L_k, \emptyset) + 4$ $\longrightarrow$ • Fact from appendix

$= \sum_{k=1}^{2} 2 \cdot c_1(L_k) + 4$

$> 0 \iff 2c_1(L_1) + 2c_1(L_2) > -4$

$\iff c_1(L_1) + c_1(L_2) > -2$

$\iff c_1(L_1) > -1 \ \& \ c_1(L_2) > -1$. ∎

- Proof (2: Symplectic)
  - J integrable $\Rightarrow D_u = \bar{\partial}_J$ the Dolbeault derivative (determined by $\mathbb{C}$-struct on M)

    $\hookrightarrow$ D_u respects the splitting $u^*TM \cong \bigoplus L_k$

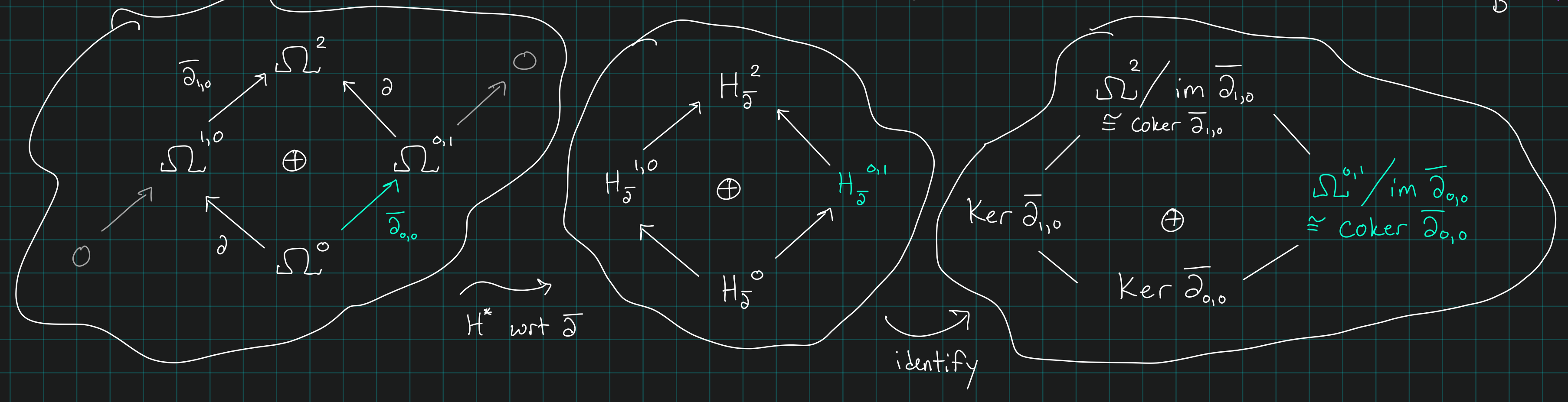
$\operatorname{Coker}(A \xrightarrow{f} B) \cong B/\operatorname{im} A$

def:

$$\begin{tikzcd} A \arrow[r] \arrow[d] & 1 \arrow[d] \\ B \arrow[r] & B \coprod_A 1 \end{tikzcd}$$

terminal

  - $\operatorname{coker}\left(\Omega^0(\Sigma; L_k) \xrightarrow{D_u = \bar{\partial}_J} \Omega^{0,1}(\Sigma; L_k)\right) = H^{0,1}_{\bar{\partial}_J}(\Sigma; L_k)$



- Surjective $\iff$ coker $= 0$, so STS $c_1(L) \geq -1 \Rightarrow H^{0,1}_{\bar{\partial}}(\Sigma; L) = 0$.
- Kodaira-Serre duality: $H^{0,1}_{\bar{\partial}}(\Sigma; L) \xrightarrow{\sim} H^{1,0}_{\bar{\partial}}(\Sigma; L^\vee \otimes K_\Sigma)^\vee$, $K_\Sigma = \Omega^2_\Sigma = \Lambda^{\dim \Sigma} \Omega^1_\Sigma$
- Kodaira vanishing: If $\begin{matrix}\mathbb{C} \to \mathcal{L} \\ \downarrow \\ \Sigma\end{matrix}$ is a positive $\overset{\text{holo}}{\vee}$ line bundle, $H^{>0}(\Sigma; \mathcal{L} \otimes K_\Sigma) = 0$

Use that $c_1(\mathcal{L}) =$ self intersection # of the zero section, "positivity of intersections"

$\Rightarrow \mathcal{L}$ has nonzero holo. sections $\iff c_1(\mathcal{L}) \geq 0$

- Set $\mathcal{L} := L^\vee \otimes K_\Sigma$, then $c_1(L^\vee \otimes K_\Sigma) < 0 \iff$ no nonzero holo. sections

$\overset{?}{\Rightarrow} H^{1,0}_{\bar{\partial}}(\Sigma; L^\vee \otimes K_\Sigma)^\vee = 0$

- Compute $c_1(L^\vee \otimes K_\Sigma) = c_1(L^\vee) + c_1(K_\Sigma)$

$= c_1(L^\vee) - c_1(T\Sigma)$

$= c_1(L^\vee) - e(TS^2)$ since c_1 is a top class

$= c_1(L^\vee) - (1 + (-1)^2)$

$= c_1(L^\vee) - 2$

$= -c_1(L) - 2$

So $c_1(L^\vee \otimes K_\Sigma) < 0 \iff -c_1(L) - 2 < 0$

$\iff -c_1(L) < 2$

$\iff c_1(L) > -2$

$\iff c_1(L) \geq -1$. □

3.3: Regularity

- **Lemma (3.3.3)**: Let (M^4, J) be almost complex and $u : \mathbb{CP}^1 \to M$ an immersed J-holomorphic sphere. Then D_u is surj. $\iff c_1(u^* TM) \geq -1$.

  - Proof
    - If ζ is a vector field on $\Sigma \Rightarrow D_u(du \circ \zeta) = du \circ \bar{\partial}_J \zeta$
    - u an immersion $\Rightarrow$ For $L_0 := \operatorname{im} du \subseteq u^* TM$, $D_u(L_0) \subseteq L_0$
    - Pick a Hermitian metric on $u^* TM$, set $L_1 := L_0^\perp$ so $u^* TM = L_0 \oplus L_1$

      $\hookrightarrow$ $c_1(L_0) = 2$, $c_1(L_1) = c_1(u^* TM) - c_1(L_0)$
      $$= c_1(u^* TM) - 2$$
    - $c_1(L_0) \geq -1$: okay!

      $c_1(L_1) \geq -1 \iff c_1(u^* TM) - 2 > -1$
      $$\iff c_1(u^* TM) > +1. \quad \blacksquare$$

- **Main Thm (3.3.4)**: Let $(\underbrace{\Sigma}_{\text{sphere}}, j_\Sigma) \hookrightarrow (M^4, J)$ be a holo. embedded sphere with $\Sigma^2 := \underbrace{\Sigma \cdot \Sigma}_{\text{self-intersection}} := p$

  Then J is regular for $A := [\Sigma] \iff p \geq -1$.

- Proof: By a prev thm (2.6.4), all $u \in \mathcal{M}(A, J)$ are embedded. Now apply lemma 3.3.3
  - Why this implies the theorem:

    - <u>Lemma (3.3.3)</u>: Let $u \in \operatorname{Imm}^\infty(\mathbb{P}^1, M)^{J\text{-hol}}$, then D_u is surjective $\iff c_1(u^* TM) \geq 1$

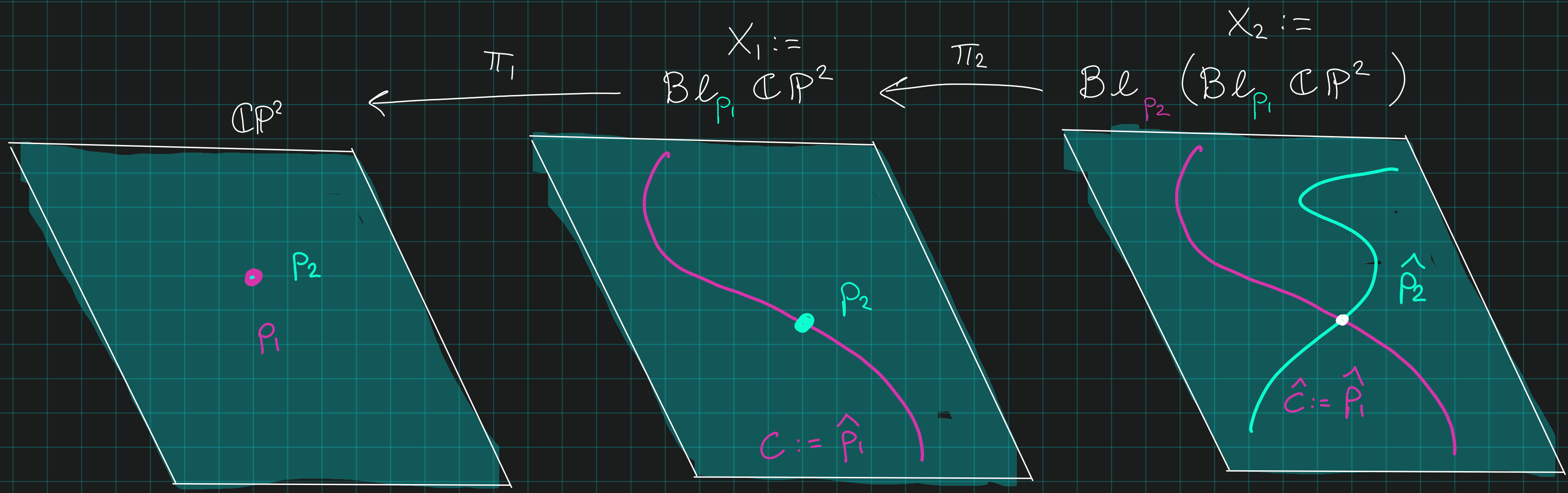
    - Σ a sphere $\Rightarrow H^*(S^2; \mathbb{Z}) = \mathbb{Z}[x]/\langle x^2 \rangle$, $|x| = 2$ $\Rightarrow \chi(\Sigma) = 1 - 0 + 1 = 2$
    - Adjunction formula: $2 - 2g + A^2 = c_1(A)$

      $g = 2 \Rightarrow A^2 = c_1(A)$

      $A = \Sigma \Rightarrow p = c_1(\Sigma)$? No idea!

An Example of Non-regularity

- Producing a non-regular embedded curve: $Bl_{p_2}(Bl_{p_1}\mathbb{CP}^2)$



- Then $C \subseteq X_1$ is regular since $c_1\left(\nu(C \hookrightarrow X_1)\right) = -1$ (normal bundle)

  but $\hat{C} \subseteq X_2$ is <u>not</u> since $c_1\left(\nu(\hat{C} \hookrightarrow X_2)\right) = -2 \lneq -1$ (need ≥ -1)