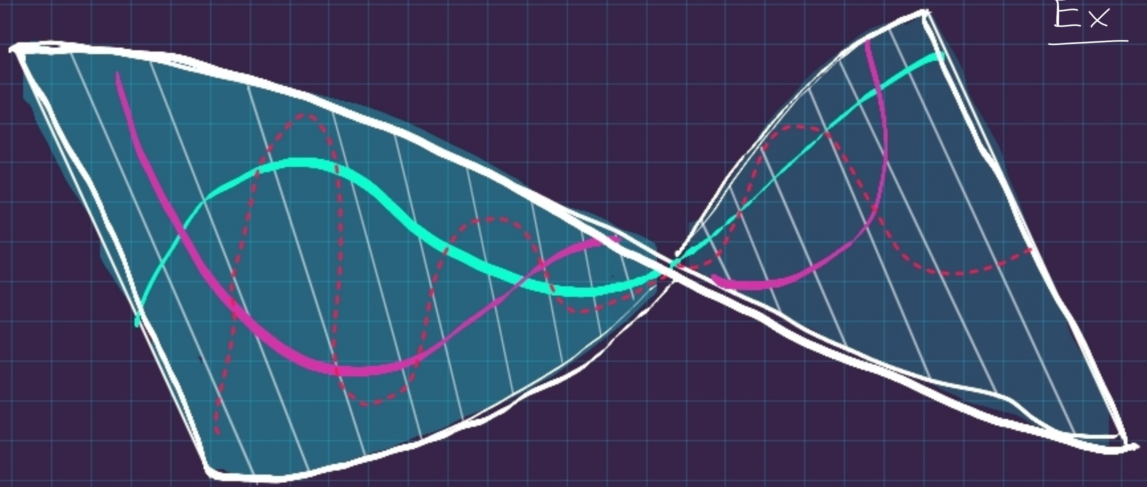
Reading Seminar:

J-Holomorphic Curves



Ex $GW^{\mathbb{CP}^n}_{L,2}(c^n, c^n) = 1$ where

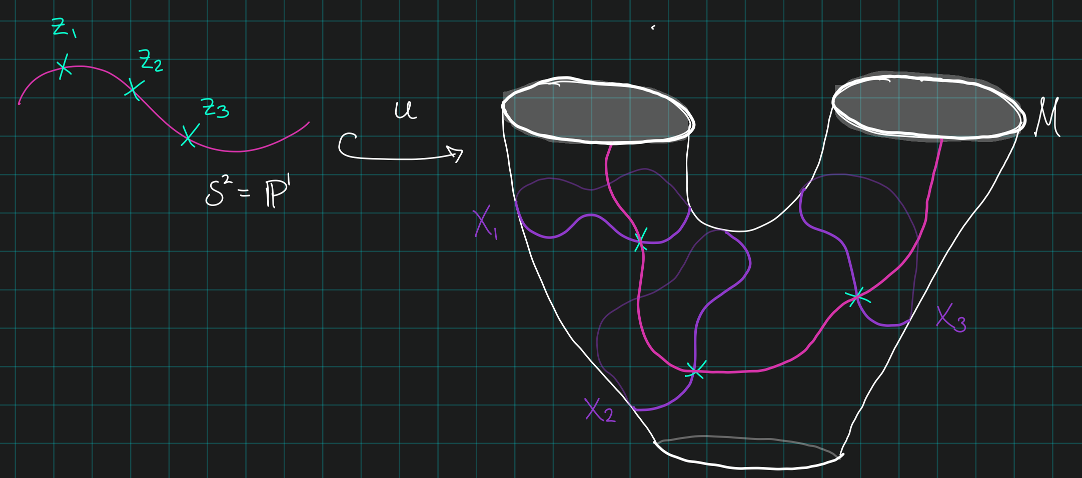
- $L := [\mathbb{CP}^1] \in H_2(\mathbb{CP}^n; \mathbb{Z})$
- c is the positive gen of $H^2(\mathbb{CP}^n; \mathbb{Z}) \cong \mathbb{Z}$
- $c^n \overset{PD}{=} pt \in H^{Top}(\mathbb{CP}^n; \mathbb{Z})$

Interpretation: Any two lines in $\mathbb{CP}^n$ meet at one point.

Goal / Outline

$GW^M_{A,k}(\alpha_1,\dots,\alpha_k) = \{[u, z_1, \dots, z_k] \in \mathcal{M}^*_{0,k}(A,J) \mid u(z_i) \in X_i\}$ where $X_i = \alpha^{PD}$

$\in H^*M$ $\quad$ = signed count of J-holo spheres passing through submfds X_i

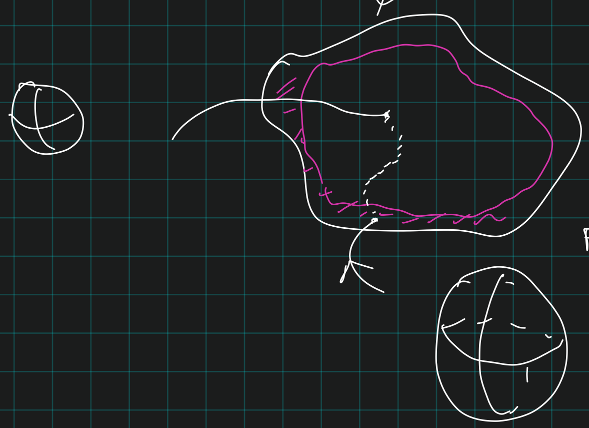


- Long-term goals:
  - Define $GW(M, A, g, n)$: Count of (isolated!) J-holomorphic spheres representing $A \in H_2(M;\mathbb{Z})$
  - Other topics
    - Applications (Ch. 8/9): Nonsqueezing, periodic orbits, obstructing Lagrangian embeddings
      Structure theorem for rational/ruled symplectic 4-mfds, $Symp(M,\omega)$ for $M = S^2\times S^2$, $\mathbb{CP}^2$, $\mathbb{R}^4$
      homotopy type
    - Ch. 11: Quantum cohomology, GW potential, Frobenius mfds
    - Ch. 12: Links to Floer homology, vortex eqns.

- Intermediate goals
  - Transversality: Show the moduli spaces are fin. dim. sm. mfds, regularity
  - Compactness: Investigate bubbling at limits of sequences ← Weakly monotone (semipositive)
  - Define the GW pseudocycle:
    - Define the moduli space of stable maps, Gromov compactness
    - Investigate combinatorics at the boundary (trees)
  - Gluing

$\underbrace{\dim \mathcal{M}^*_{0,k}(A,\Sigma,J) \overset{?}{=} 2n-6 \qquad \dim \text{Bubbles} = 4}$ $< 2n$

Choose J so no bubbles!



$E = \int_{S^2}\omega$ big?

Ch. 3: Transversality (& Regularity)

- Why regularity is important:

$\Rightarrow \dim \mathcal{M}^*_{0,k}(A,J) = 2n + 2c_1(A) + (2k-6)$ (expected dimension)

$\Rightarrow$ (+ other conditions) ev_J is a pseudocycle, can use it to calculate GW.

- If $J \in \mathcal{J}_{reg}(M,\omega)$,

6.2.6: $\mathcal{M}^*_{0,T}(B,J) :=$ simple stable maps modeled over a k-labeled tree is a sm. mfd. of dim $d = \mu(B,K) - 2e(T)$

6.6.1: ev_K is a pseudocycle.

$\hookrightarrow$ Needs semipositivity condition: no spherical homology classes A with $\begin{cases} \omega(A) > 0 \\ c_1(A) < 0 \end{cases}$

$\hookrightarrow$ Every J-holo sphere having $c_1(A) < 0$ $\Rightarrow$ boundary strata in $\overline{\mathcal{M}}^*_{0,T}(B,J)$ have dim $d_k < u(A,T)$.

- Regularity: implies D_u is surjective & $\forall$ trees, edge evals are transverse to edge diagonals (6.2.1)

$\Rightarrow \mathcal{M}^*_{0,k}(A,J)$ has expected dim & ev_J are pseudocycles

$\hookrightarrow$ To compute GW, need to recognize when $J \in \mathcal{J}_{reg}(M,\omega)$.

$\hookrightarrow$ Suffices to check on all trees T and all homology classes $\{B_\alpha\}$ in $\overline{\mathcal{M}}^*_{0,k}(A,J)$ where $A = \sum_{\alpha \in T} m_\alpha B_\alpha$ (in H_2 ...?)

Ch. 3: Transversality $\rightsquigarrow$ 3.3: Regularity Criteria

- Relevant defns:

  - $D_u : \Omega^0(\Sigma; u^*TM) \longrightarrow \Omega^{0,1}(\Sigma; u^*TM)$ where
$$\begin{tikzcd} u^*TM \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & TM \arrow[d] \\ \Sigma \arrow[r, "u"] & M \end{tikzcd}$$
[Not nec. rank 2!]

  - Transversality $\iff$ D_u surjective $\overset{\forall u}{\iff}$ J is regular for A

- Recall the main results:

3.3.1) J integrable, $\mathbb{CP}^1 \xrightarrow{u} M$, $u^*TM \cong \oplus \mathcal{L}_k$ with $c_1(\mathcal{L}_k) \geq -1 \;\; \forall k \implies$ regular

3.3.2) Generalize to almost-complex mfds:

$\begin{array}{c} \mathcal{E} \\ \downarrow \\ \mathbb{CP}^1 \end{array}$ rank n, D any CR operator, $\mathcal{E} \cong \oplus \mathcal{L}_k$ all D-invariant, then

$\left(c_1(\mathcal{L}_k) \geq -1 \;\; \forall k \iff \text{regular} \right)$

3.3.3) J almost complex, $\dim_{\mathbb{R}} M = 4$, $\mathbb{CP}^1 \overset{u}{\hookrightarrow} M$ immersed, then $c_1(u^*TM) \geq +1 \iff$ regular

Thms

3.3.4) J almost complex, $\dim_{\mathbb{R}} M = 4$, $\mathbb{CP}^1 \overset{u}{\hookrightarrow} M$ embedded, then $u(\mathbb{CP}^1) \cdot u(\mathbb{CP}^1) \geq -1 \iff$ regular for $A := [u(\mathbb{CP}^1)]$

3.3.5) $\widetilde{M} := S^2 \times M$, $\widetilde{A} := [S^2 \times pt]$, then $\forall J \in \mathcal{J}(M, \omega)$, $i \times J$ is regular for $\widetilde{A}$.

any symp. mfd

Ch. 3: Transversality $\rightsquigarrow$ 3.3: Regularity Criteria

- **Lemma (3.3.3):** Let (M^4, J) be almost complex and $u : \mathbb{CP}^1 \to M$ an immersed J-holomorphic sphere. Then D_u is surj. $\iff$ $c_1(u^*TM) \geq -1$.

- Proof
  - If ζ is a vector field on Σ $\Rightarrow$ $D_u(du \circ \zeta) = du \circ \bar{\partial}_J \zeta$
  - u an immersion $\Rightarrow$ For $L_0 := \operatorname{im} du \subseteq u^*TM$, $D_u(L_0) \subseteq L_0$
  - Pick a Hermitian metric on u^*TM, set $L_1 := L_0^{\perp}$ so $u^*TM = L_0 \oplus L_1$

    $\hookrightarrow$ $c_1(L_0) = 2$, $c_1(L_1) = c_1(u^*TM) - c_1(L_0)$

    $= c_1(u^*TM) - 2$
  - $c_1(L_0) \geq -1$: okay!

    $c_1(L_1) \geq -1 \iff c_1(u^*TM) - 2 > -1$

    $\iff c_1(u^*TM) > +1$. $\blacksquare$

- **Main Thm (3.3.4):** Let $(\overbrace{\Sigma}^{\text{sphere}}, j_\Sigma) \hookrightarrow (M^4, J)$ be a holo. embedded sphere with $\overbrace{\Sigma^2 := \Sigma \cdot \Sigma}^{\text{self-intersection}} := p$

  Then J is regular for $A := [\Sigma]$ $\iff$ $p \geq -1$.

- Proof: By a prev thm (2.6.4), all $u \in \mathcal{M}(A, J)$ are embedded. Now apply lemma 3.3.3
  - Why this implies the theorem:

    - **Lemma (3.3.3):** Let $u \in \mathrm{Imm}^{\infty}(\mathbb{P}^1, M)^{J\text{-hol}}$, then D_u is surjective $\iff$ $c_1(u^*TM)$

    - Σ a sphere $\Rightarrow$ $H^*(S^2; \mathbb{Z}) = \mathbb{Z}[x]/\langle x^2 \rangle$, $|x| = 2$ $\Rightarrow$ $\chi(\Sigma) = 1 - 0 + 1 = 2$
    - Adjunction formula: $2 - 2g + A^2 = c_1(A)$

      $g = 0 \Rightarrow A^2 + 2 = c_1(A)$

      $\underbrace{c_1(A) > +1}$ $\iff$ $A^2 + 2 > +1$

      ? $\iff A^2 > -1$?

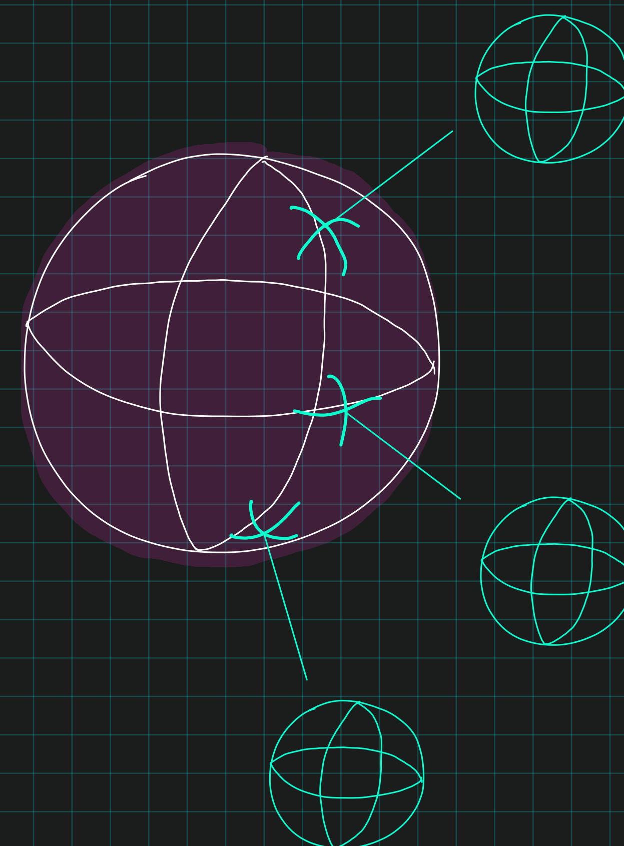
?

Ch. 3: Transversality $\rightsquigarrow$ 3.3: Regularity Criteria

Eg. 3.3.6: Regular vs non-regular curves

· Regular curves

- Set $M := S^2 \times S^2$ with $J := j \times j$



- Use universal property of product:

$$\begin{tikzcd}
 & & S^2 \\
S^2 \arrow[r, "\forall u"] \arrow[urr, bend left, "u_1"] \arrow[drr, bend right, "u_2"'] & S^2 \times S^2 \arrow[ur, "\pi_1"] \arrow[dr, "\pi_2"'] & \\
 & & S^2
\end{tikzcd}$$

∴ Holomorphic components

So $u(z) = [u_1(z), u_2(z)]$

Projections are holomorphic

- $u_i : S^2 \to S^2$ has degree $d_i \geq 0$ (Why?)

- $u^* TM \cong u_1^* TM \oplus u_2^* TM := L_1 \oplus L_2$

- $\deg(u_i^* TM) = 2 \cdot d_i \geq 0$

These curves are regular using 3.3.1: $c_1(L_k) \geq -1$ for every summand of $u^* TM$ for J-hol spheres.

· Upshot: Any curve $C \overset{u}{\hookrightarrow} S^2 \times S^2$ is regular for $J := j \times j$

↳ Need other ω-tame J to produce non-regular curves.

Ch. 3: Transversality $\rightsquigarrow$ 3.3: Regularity Criteria

- **Non-regular curves:** Can construct a symp. mfd $M := (S^2 \times S^2, \omega^\lambda)$, J an ω^λ compat., $\overline{\Delta} := \{(z, -z) \mid z \in S^2\}$ non-regular

  $A(z) = -z$

- Identify $S^2 \to S^2 \cong \mathbb{P}(L \oplus \mathbb{C}^1)$ where L is a deg $d = 2k > 0$ holo. line bundle.

  $J_0 := j \times j$

  Let J_{2k} be the complex structures

- $L \subseteq \mathbb{P}(L \oplus \mathbb{C}^1)$ as a sub-bundle corresponds to a section $C_L \in \Gamma(\mathbb{P}(L \oplus \mathbb{C}^1))$ with normal bundle L^*
- So $k > 0 \Rightarrow \boxed{c_1(L^*) < -1} \Rightarrow C_L$ not regular.

- Push Kähler form through correspondence to get a symp form

$$\mathbb{P}(L \oplus \mathbb{C}^1) \longrightarrow S^2 \times S^2$$

$$\kappa \longmapsto \omega^\lambda = \lambda \cdot \pi_1^*(\sigma) \oplus \pi_2^*(\sigma),$$

σ an area form on S^2

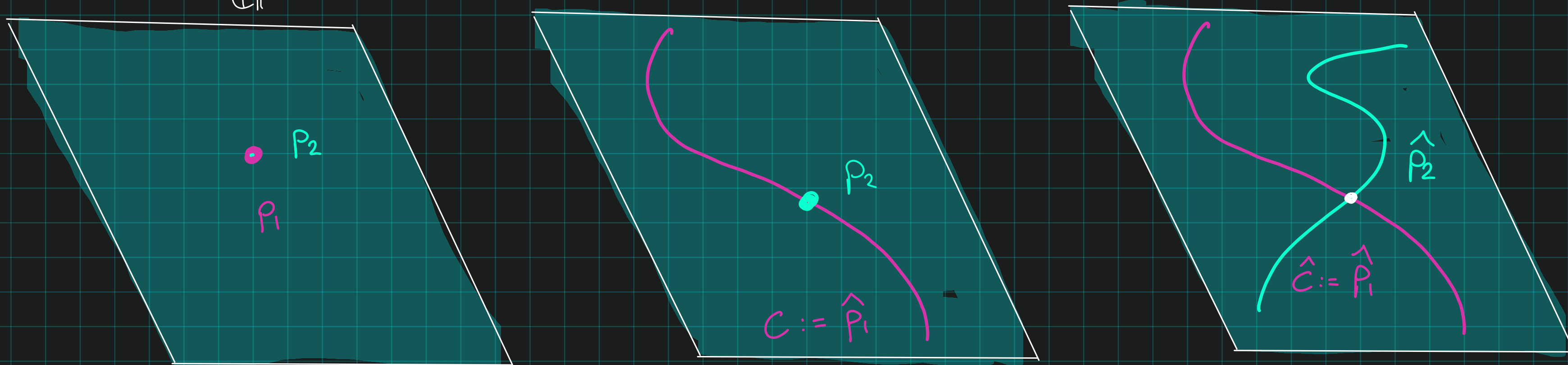
$\lambda > 1$

- **Then $(S^2 \times S^2, \omega^\lambda)$ non-regular for J_{2k}** (when $\lambda > 1$)

An Example of Non-regularity

- Producing a non-regular embedded curve: $Bl_{p_2}(Bl_{p_1}\mathbb{CP}^2)$

$$\mathbb{CP}^2 \xleftarrow{\pi_1} X_1 := Bl_{p_1}\mathbb{CP}^2 \xleftarrow{\pi_2} X_2 := Bl_{p_2}(Bl_{p_1}\mathbb{CP}^2)$$



- Then $C \subseteq X_1$ is regular since $c_1(\nu(C \hookrightarrow X_1)) = -1$ (normal bundle)

  but $\hat{C} \subseteq X_2$ is <u>not</u> since $c_1(\nu(\hat{C} \hookrightarrow X_2)) = -2 \leq -1$ (need ≥ -1)

3.4: Constrained Curves

- Fix A, fix Σ (not necessarily connected) $\Rightarrow \mathcal{M}^*(A,\Sigma,J)$ is the space of simple maps with smooth disconnected domains.

- Fix $\vec{\omega} = [\omega_1, \cdots, \omega_n] \in \mathrm{Sym}^n \Sigma$, define $ev_{\vec{\omega}} : \mathcal{M}^*(A,\Sigma,J) \to M^{\times n}$

$$u \mapsto [u(\omega_1), \cdots, u(\omega_n)]$$

and set $\mathcal{M}^*(A,\Sigma,J,\vec{\omega},X) := \{ u \in \mathcal{M}^* \mid ev_{\vec{\omega}}(u) \in X \}$

$X \subseteq M^n$ a submfd

- Say J is regular $\Leftrightarrow$ $ev_{\vec{\omega}}$ is transverse to X, write $\mathcal{J}_{reg}(A,\Sigma,\vec{\omega},X)$.

- Thm (3.4.2)

Technical

Every $\vec{x} \in M^{\times n}$ is a regular value for $\widetilde{ev}_{\vec{\omega}}$.

↳ Needed in important examples where $X \cap \Delta \neq \emptyset$. Technical proof!!!

- Thm (3.4.1)

$\mathcal{J}_{reg}(A,\Sigma,\vec{\omega},X) \subseteq \mathcal{J}$ is Baire 2nd class, and $\dim \mathcal{M}^*(A,\Sigma,\vec{\omega},X) = 2n + 2c_1(A) - \mathrm{codim}\, X$.

↳ Idea for pf: $\widetilde{ev}_{\vec{\omega}} : \mathcal{M}^*(A,\Sigma,\mathcal{J},\vec{\omega},X) \to M^{\times n}$ $\Rightarrow d\,\widetilde{ev}_{\vec{\omega}}$ is surjective @ all pts

3.4.2 $\Rightarrow$ $\widetilde{ev}_{\vec{\omega}}$ transverse to every $N \subseteq M^{\times n}$

$\Rightarrow \widetilde{\mathcal{M}}^*$ is a $C^{\ell-1}$ Banach mfd (taking $\mathcal{J}^\ell$ in $\widetilde{\mathcal{M}}^*$)

$$\begin{array}{c} \widetilde{\mathcal{M}}^* \\ \shortparallel \\ \mathcal{M}^*(A,\Sigma,\mathcal{J}^\ell,\vec{\omega},X) \\ \downarrow \\ \mathcal{J}^\ell \end{array}$$

Now show $\mathcal{M}^*(A,\Sigma,\mathcal{J}^\ell,\vec{\omega},X) \to \mathcal{J}^\ell$ is Fredholm, has expected index, apply Sard-Smale $\underbrace{+\ \varepsilon}_{\text{Taubes' arg.}}$

↳ Slogan: generically, $ev_{\vec{\omega}} \pitchfork \Delta_{M^n}$ implies moduli of distinct intersecting spheres is a sm. mfd of expected dimension (for general J)

↳ Needed for gluing

3.4: Constrained Curves

- "Thick" diagonal $\Delta^n := \{ \vec{w} \in M^n \mid w_i = w_j \text{ for some } i,j \}$. Note $(M^n \setminus \Delta^n) = \mathrm{Conf}_n(M)$
  - ↳ Used in proof

- Need later:

$$\left.\begin{aligned} ev_{\vec{w}} &: \mathcal{M}^*(A, \Sigma, J) \to M^{\times n} \\ ev'_{\vec{w}} &: \mathcal{M}^*(A', \Sigma', J') \to M^{\times n} \end{aligned}\right\} \text{ Need to intersect transversally in } M^{\times n}$$

- Used in a pf: $Sp(M)_0 \curvearrowright M_0$ transitively on each component

$$\left.\begin{aligned} &Sp(M) \curvearrowright \mathcal{M}^*(A, \Sigma, \mathcal{J}^\ell) \\ &\psi \mapsto \big((u, J) \mapsto (\psi^{-1} u, \psi^* J) \end{aligned}\right\} \widetilde{ev_{\vec{w}}} \text{ is equivariant for these!}$$

- 3.5 preview: If $u \approx$ J-holomorphic, it can be perturbed if D_u is surjective with a unif bounded right-inverse.

  $\| \bar{\partial}_J(u) \|_{L^p} < \varepsilon$.

Skip 3.4 (Long proof)

3.5: Implicit fn thm?: Appx J-hol curves

Ch. 4 next week